THE MATHEMATICAL GAZETTE

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc., F.R.S.
AND
PROF. E. T. WHITTAKER, Sc.D., F.R.S.

LONDON
G. BELL & SONS, LTD., PORTUGAL STREET, KINGSWAY, W.C. 2.
AND BOMBAY

Vol. XV., No. 206. MARCH, 1930. 3s. 6d. Net.

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Intending members are requested to communicate with one of the Secretaries. The subscription to the Association is 15s. per annum, and is due on Jan. 1st. It includes the subscription to "The Mathematical Gazette."

Change of Address should be notified to a Secretary. If Copies of the "Gazette" fail for lack of such notification to reach a member, duplicate copies can be supplied only at the published price.

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The Mathematical Association.

THE Annual Meeting of the Mathematical Association was held at the London Day Training College, on 6th-7th January, 1930.

On Monday, 6th January, the proceedings opened at 2.30 p.m., with the transaction of formal business, Dr. Sheppard being in the chair. About 60 members were present.

The Report of the Council * for 1929 was considered and accepted. The Treasurer explained that, as usual, it was not possible so early in the year to issue a complete audited statement of the accounts of the previous year. He outlined the financial position, which was very satisfactory, and promised that a complete statement should be issued in due course in the *Gazette*.

The Chairman announced that the Council had the pleasure of nominating Prof. A. S. Eddington, D.Sc., F.R.S., for the office of President for the year 1930. He was elected by acclamation.

The Officers and Council were re-elected for 1930, with the exception of Miss M. J. Griffiths and Mr. F. G. Hall, who retired and were not eligible for re-election. Miss R. H. King and Mr. C. J. A. Trimble were elected in their places.

The Special Report of the Council on the relation of the Branches to the main Association was considered and discussed, being proposed by Miss Punnett and seconded by Mr. Katz.

The decisions of the Council will be found in the account of the Special General Meeting (p. 34), which was held to consider the suggested alterations of the rules.

Mr. Boon expressed the thanks of the meeting to the Council for its sympathy with the wishes of members in the matter of the relations between the Branches and the Association, and especially to Dr. Sheppard for his work, carried on as it had been under the handicap of severe illness.

* v. pp. 36, 37.

The President took the chair in the afternoon at 4 p.m., and called upon Mr. B. L. Gimson, B.Sc., to open the discussion on *The Arithmetic of Citizenship.** This was followed after the usual interval by a lecture on *The Use of Spherical Harmonic Functions in Mathematical Physics*,† delivered by Prof. S. Chapman, D.Sc., F.R.S.

On Tuesday, 7th January, at 10.0 a.m., Mr. G. W. Spriggs, M.Sc., F.C.P., opened a discussion on *Problems of Individual Education, with special reference to work in Mathematics*.‡

After an interval Prof. W. M. Roberts, M.A., gave a lecture on *Gunnery and some of its Mathematical Problems.*†

The afternoon sitting opened at 2.30 p.m. with Dr. Sheppard's Presidential Address on *Mathematics for Study of Frequency Statistics*.†

After the interval a discussion was opened by Miss Hilda Hudson, O.B.E., Sc.D., on *The Mathematician in Ordinary Intercourse.* §

The usual votes of thanks brought the proceedings to a close.

A Publishers' Exhibition was arranged for the two days, and was open on Tuesday till 7 p.m.

THE SPECIAL GENERAL MEETING.

THE Special General Meeting of the Association was held at Bedford College at 2.30 p.m. on Saturday, 1st February, 1930, to consider the proposed alterations in the Rules. These were as stated in the following quotation from the summons. There were present 22 members. Prof. Lodge took the chair.

At the recent Annual Meeting of the Association the following decisions, put forward by the Council, were reached:

1. That every Branch in the United Kingdom which includes in its membership not fewer than 25 members of the Association shall be entitled to have a representative on the Council, the representative to be a member both of the Branch and of the Association and to be appointed by the Branch.
2. That the limitation (under Rule VI) by which the total annual payment by the Association to any one Branch shall not exceed four pounds shall be removed.

Moreover, the Council at its Autumn Meeting decided to propose a modification of the rules so that the chairmen of the three Teaching Committees should be *ex officio* members of the Council.

Accordingly, to put these decisions into effect, this Special General Meeting is being called to consider the following proposed alterations of rules:

RULE VI (LOCAL BRANCHES).

The last sentence of the last paragraph of this rule—("The total annual payment to any one Branch under this rule shall not exceed four pounds")—to be omitted.

RULE VII (OFFICERS AND COUNCIL).

The following paragraphs to be substituted for the second and third paragraphs of this rule:

The business of the Association shall be conducted by a Council, consisting of the Officers, a representative of each local Branch in the United

* *v.* pp. 55-61. † Will be published later. ‡ *v.* pp. 38-54. § pp. 62-67.

Kingdom which includes in its membership not fewer than 25 members of the Association, the chairmen (*ex officio*) of the three Teaching Committees, the Honorary Secretary (*ex officio*) of the General Teaching Committee, and other members, five to form a quorum. The number of Officers and unofficial members of the Council, other than the President, the Vice-Presidents, the representatives of local Branches, and the chairmen of the Teaching Committees, shall not exceed seventeen, of whom ten at least shall be persons residing within easy access of London.

Each representative of a local Branch shall be a member both of the Branch and of the Association and shall be appointed by the Branch. The Officers and other members of the Council, with the exception of the chairmen of the Teaching Committees and the Honorary Secretary of the General Teaching Committee, shall be elected each year at the Annual General Meeting.

The sixth paragraph of Rule VII (giving the Council power to co-opt a representative of a local Branch) to be omitted.

Rule XIV (Annual General Meeting).

The following paragraph to be substituted for the fourth paragraph of this rule:

The Officers, the Auditor and the other members of the Council, with the exception of the representatives of local Branches, the chairmen of the Teaching Committees and the Honorary Secretary of the General Teaching Committee, shall be elected. (*Rule VII.*)

C. PENDLEBURY, } Hon.
M. PUNNETT, } Secs.

The alterations in the Rules were put to the meeting *seriatim*. They were all passed unanimously and without any modifications.

GLEANINGS FAR AND NEAR.

727. There are some men who are counted great because they represent the actuality of their own age, and mirror it as it is. Such an one was Voltaire, of whom it was epigrammatically said, "he expressed everybody's thoughts better than anybody." But there are other men who attain greatness because they embody the potentiality of their own day and magically reflect the future. They express the thoughts which will be everybody's two or three centuries after them. Such an one was Descartes.—T. E. Huxley, "On Descartes' 'Discourse,'" *Collected Essays*, i. 1893, p. 167.

728. Il a fallu introduire des signes nouveaux; j'ai donné une attention particulière a cet objet, persuadé que le secret de la puissance de l'Analyse consiste de la choix et l'emploi heureux des signes simples et caractéristiques de la chose qu'ils doivent représenter. Je me suis prescrit à cet égard les règles suivantes: (1) de rendre les notations le plus qu'il étoit possible analogues à des notations reçues; (2) de ne point introduire des notations inutiles et que j'aurois pu remplacer sans confusion par des notations déjà en usage; (3) de les choisir très-simples, en y faisant entrer cependant toutes les variétés qu'exigoient les différences des opérations.—Arbogast, *Du Calcul des Dérivations*, 1800, *Préface*, p. ii, 1800. [Per Dr. G. J. Lidstone.]

729. The forcible language and striking illustrations by which those who are past hope of being even beginners [in science] are prevented from becoming conscious of intellectual exhaustion before the hour has elapsed.—J. C. Maxwell. [Quoted by Tait, N. 21, p. 320.]

REPORT OF THE COUNCIL FOR THE YEAR 1929.

During the year 1929, 73 new members of the Association have been admitted. The number of members now on the Roll is 1223: of these, 7 are Honorary Members, 61 are Life Members by composition, 17 are Life Members under the old rule, and 1138 are ordinary members.

The Council regret to have to record the deaths of Mr. W. B. Allcock, Fellow of Emmanuel College, Cambridge; Mr. J. A. Blaikie, formerly of the Scottish Education Department and Fellow of Caius College, Cambridge; Mr. T. J. I'Anson Bromwich, Sc.D., F.R.S., of St. John's College, Cambridge, sometime Professor of Mathematics in Queen's College, Galway; Mrs. Winifred Cranko; Miss J. M. Earle, Head Mistress of the Girls' High School, Tamworth; Mr. F. M. Gilbert, of Framlingham College, Suffolk; Professor P. G. Gundry, Ph.D., of the Transvaal University College, Pretoria; Mr. F. W. Haselfoot; Emeritus Professor M. J. M. Hill, LL.D., Sc.D., F.R.S., formerly Professor of Mathematics in the University of London, and President of the Mathematical Association for the years 1926 and 1927; Mr. A. E. Holme, of Skipton-in-Craven; Mr. T. W. Hope, formerly of St. Olave's School, Southwark; and Dr. W. L. Mollison, Master of Clare College, Cambridge.

At the Annual Meeting of the Liverpool Mathematical Society, held at the University of Liverpool on 27th May, 1929, it was decided to apply for recognition of the Society as a local branch of the Mathematical Association. The Council very cordially welcome this new branch.

There are now twelve Local Branches of the Association—nine in England and Wales, and three in Australia. The relation of the Branches to the parent body has been under consideration by the Council during the year, and a special report on the subject is being presented for consideration at the Annual General Meeting.

The Teaching Committees have been engaged during the year on a revision of the "Report on the Teaching of Mathematics in Girls' Schools," and on a new "Report on the Teaching of Mechanics." The former was distributed to members of the Association with the December issue of the *Gazette*. The latter is now in the printers' hands. A third edition of the Geometry Report of 1923 was called for in April.

In accordance with the Rules of the Association the Teaching Committees, which have held office since 1926, have recently been reconstituted. The new Committees will enter upon their duties immediately, and will remain in office for four years, until January 1934. Much of the hardest and most valuable work of the Association is done by the Teaching Committees, and the Council takes this opportunity of offering to the retiring Committees the thanks of

members for what they have accomplished during their period of office.

The Problem Bureau, which was started in 1928 with the object of assisting members of the Association in the solution of problems, has had an active year and clearly satisfies a genuine need. The most cordial thanks of the Association are due to Mr. A. S. Gosset Tanner of Derby School, and his coadjutors, Mr. S. J. Tupper and others, for their very ready and willing help in time of trouble.

The Library has grown comparatively little during the year, but members have made much more use of it than in any one year previously. A Second List of contents, which includes the books given by Mrs. Charles Godfrey, was issued in May, and a Third List, comprising chiefly the numerous offprints and papers left by Prof. R. W. Genese, is in preparation. The thanks of members are again due to Mr. F. Beames for the work done throughout the year and especially for ensuring that members did not suffer on account of the Librarian's absence in the summer.

Dr. W. F. Sheppard retires after this meeting from the office of President, which he has held for two years, and the Council desire to express their most grateful thanks to him for the services which he has rendered to the Association. The knowledge of the inner life of the Association which he had acquired during his twenty-four years of membership makes those services of special value. The Council feel sure also that members will sympathise with Dr. Sheppard in having his year of office interrupted by a serious operation and trust that his recovery may be complete and permanent.

The Council nominate Professor A. S. Eddington, D.Sc., F.R.S., to be President for the year 1930, and Dr. W. F. Sheppard to be a Vice-President.

The Council desire to express their thanks to Miss M. J. Griffith and to Mr. F. G. Hall, who now retire from the Council, for the services which they have rendered to the Association during their tenure of office as members of the Council since 1923 and 1925 respectively.

The Council desire again to acknowledge the indebtedness of the Association to Mr. W. J. Greenstreet for the continuance of his work as Editor of the *Mathematical Gazette*. At the beginning of the year 1929 Mr. Greenstreet had completed thirty years of service as sole Editor; and it was decided by some friends to show their appreciation of his merits, both as a man and as an editor, by offering to him a commemorative gift. The Council learned with pleasure of the proposal, and the resources of the Association have been used to give it publicity. The fund, for which Mr. Pendlebury is acting as Treasurer, still remains open, so that anyone who learns of it for the first time from this Report may have an opportunity of contributing, but it will be closed early in 1930.

PROBLEMS OF INDIVIDUAL EDUCATION, WITH SPECIAL REFERENCE TO WORK IN MATHEMATICS.

By G. W. Spriggs, M.Sc., F.C.P.

At the outset I desire to express my deep appreciation of the honour of being invited to lay before your Association some of the investigations into educational practice which I have undertaken during the past few years. I ask only that they be regarded as any scientific experiment, as a search for some true principles, and that at any stage of that quest failure was to be welcomed as warmly as success, as affording suggestions for further enquiries. That the experiment has been attended with a measure of success does not make me ready to proclaim it as the *ne plus ultra*: I submit the facts to your judgment in the hope that they may afford material for a profitable discussion and stimulate some abler investigators to pursue the matter further.

As teachers of Mathematics it behoves us to assess the obligations and possibilities of our calling in their full breadth, more so, perhaps, in view of the dangers arising from the logical structure of our subject, than is necessary with any other subject of the curriculum. For, while it is our duty to impart a competent knowledge of Mathematics, that is but one aspect of our wider task—the effective education of our charges. We are thus confronted with the important problem, how, without sacrificing efficiency in the acquisition of knowledge by the pupil, can we frame a procedure by which it may also be made to serve his highest requirements, and, while making him more of a mathematician, may certainly make him yet more of a man? In this we must remember that logic is not life, however important a part it may play in effective living: it is an instrument to be applied to the phenomena of experience, whereby these are wrought into experiences of a new significance, having enriched potentialities in relation to the emergence of that uniqueness of the individuality which constitutes the self.

We hear much to-day of psychology in relation to education, but experience and observation tend to show that the fundamental principles of educational practice are less psychological than psycho-pathological. For traditional practice starts with the assumption that something is wrong which must be righted—usually without adequate diagnosis—by methods homœopathic or surgical according to the temperament of the practitioner.

A horticulturist wisely and profitably recognises different strains, but he does not attribute the inability of a potato to produce grapes to some form of original sin in the plant. His direct action is comparatively little directed towards the seed as such: having it, it is what it is: it is its self, rich with potentialities of achieving that self in change. He will do much to ensure that the environment is appropriate to such progressive change, and will labour to purge it of harmful influences, but in the last resource he depends upon the vigour of inherent qualities to respond to and develop in accordance with that environment. It is not wholly apposite to emphasise the difficulties arising from the child's psychological characters: many of those are of our own making—they are the response to the environment. Nor is it relevant to observe that the tree is pruned, for the vigour thus diverted is allowed to manifest itself, whereas the restrictions of educational practice tend to inhibit its very development, or, even worse, to divert it into unwholesome channels.

There has recently been a growing recognition of these facts in many quarters, one of the most noteworthy developments arising therefrom being the Dalton Plan, of which I have been asked to give an account from first-hand experience preparatory to dealing with my own practice.

Miss Parkhurst, the originator of the Plan, put her scheme into practice at Dalton High School in February 1920. It is impossible to pay adequate tribute to this pioneer in the education of the individual. Her great aim was to break the shackles of class-teaching and the domination of the teacher, and so make provision for the undeniable differences encountered in individual pupils.

To this end class-teaching gave place to organised private study, in which the pupil was made the primary, responsible and effective agent. Instead of lesson topics, upon which the teacher gives instruction and sets exercises leading to further similar lessons at later times according to time-table, a large programme of work is envisaged, and the pupil is free to organise his activities within this range in order to gain the necessary mastery of it.

At this point we meet the assignment, which is the programme of work prescribed by the instructor for a month. It contains an outline of the work, and is the key to the situation : all further details of organisation are directed to its effective achievement. In Miss Parkhurst's own words " It is not too much to say that the Dalton Laboratory Plan hinges upon the assignments ; for on the degree of skill and understanding with which they are compiled, its successful application will depend."

The assignment must present a clear perspective so that the pupil may see whither his activities are tending : it should contain helpful suggestions or " interest pockets " to stimulate him : and must be reasonable in respect of the amount of work demanded, so that he may look forward to a manageable task, which, while calling forth his best endeavours, may be successfully accomplished.

Recognition of individual differences leads to a modification of the assignment to bring it within the reach of three categories : the minimum assignment giving essentials of a sound groundwork for the weakest : a medium assignment appropriate to the moderately able : and the maximum assignment giving scope for the best pupils.

Each pupil receives a copy of the assignment at the beginning of the month, and contracts to undertake the prescribed work. The whole month's task is divided into four weekly allotments, which are further subdivided into five or six work units for purposes of record. How and when these are undertaken is henceforward the pupil's own concern. The time-table has been abandoned, and there are no special periods set aside for the study of this or any other subject. The pupil is free to work at whatever subject he will whenever he pleases, provided always that he keeps to the terms of his contract—to get the all-round job done over the whole range of the curriculum.

Since no lessons are given, the class-room ceases to exist as such, and is now called a laboratory. Each room is given over to the study of a particular subject, is equipped with the necessary text-books, reference books and material, and is in charge of a specialist instructor. The pupils move freely between the different laboratories : no bells ring to mark the end of lessons. For so long as the pupil chooses to labour at a subject, he occupies a place in the appropriate laboratory and utilises its facilities, including the instructor, whom he consults as and when special difficulties make this expedient. Having exhausted his immediate interest in that subject, or feeling the more insistent demands of another, he moves to another laboratory.

It will be obvious that the position of the teacher has undergone a complete change : instead of maintaining discipline and expounding the subject, he allows the work to proceed and impose its own discipline, holding himself in constant readiness to advise or assist when the pupil makes known his special needs.

Provision is, however, made for conferences at specified times, say for half an hour at the end of a session, when pupils foregather in the laboratories for collective discussions at which common difficulties are investigated.

With pupils working so independently of each other it will be clear that the system for recording progress demands attention. For this purpose Miss Parkhurst has elaborated her graphs. The first is the pupil's graph, containing columns for the different subjects, each divided into four spaces for the weeks, and these again into five or six sections to correspond to the work units. As any work units are satisfactorily completed, a line is drawn up the column to the appropriate height, and the number of the contract day indicated at its terminus.

The instructor has his laboratory graphs bearing the names of the pupils of his Forms or Sets, opposite which he draws lines to indicate work units and the day upon which they are completed.

Such in very brief outline are the essentials of the Dalton Plan. Experience of its working has revealed a number of problems that must be faced in its application to our schools. Its wholesale adoption with more enthusiasm than discretion is likely to give rise to most of these, and to any who are sympathetic towards its aims I would suggest that, rather than taking it as a sacrosanct revelation to be adopted in its entirety, it be regarded as a sincere indication of possibilities, and that actual working conditions be explored in the endeavour to discover a technique appropriate to these which shall realise those great ends.

In the first place the numbers of our pupils present a formidable difficulty. Under existing regulations it is a matter of simple arithmetic to see that a master will be in charge of some four or five Forms, numbering from 120 to 150 boys. The requisite individual attention cannot be given to more than about 80 boys working under the conditions of the Plan, and it is impossible to secure this in the average Secondary School.

This loss of contact suggests a practical remedy in requiring more uniform progress among the boys of a Form, and with us this took the form of requiring that a month's work should be done in a month, and later a week's work in a week, a progressive reduction of the boy's freedom to do his work as and when he will, which amounts to a serious modification of the spirit of the Plan.

Moreover, the boy's time now being entirely available for his own activity, it is likely that there will be an increase in the amount of written work to be examined by the instructor, a danger aggravated by the conscientious master who strives to safeguard his subject by making heavy demands in his assignments. This increased amount of marking is a further burden, which may react unfavourably on the effectiveness of or the opportunities for giving the requisite individual attention.

Actually we found it desirable to give more explicit guidance to the boy in the management of his affairs by reverting to a fixed time-table, so that all the boys of a Form were working at the same subject in the same room at the same time. The periods were longer than is customary, and the master was free to give such collective instruction as he felt necessary, it still being a cardinal principle of our practice that this should be reduced to a minimum, and the greatest opportunity given to the boys to pursue their activities without interruption.

If the assignment is regarded as a means of organising the work, it is merely a substitute for the more or less detailed annotated syllabus and sets of exercises in common use. But, if it is to be made serviceable in meeting individual differences, it presents great difficulties. It is possible to anticipate some fairly universal troublesome points, but to meet the numerous difficulties arising from admitted individual differences is impossible without making the assignment irrelevant in a large measure to everybody.

Yet another practical problem is to be found in the text-books available to the pupil in seeking guidance in his task. Few such books meet the psychological needs of the situation: they are primarily concerned with the subject

itself, and therefore with the logical aspect. But learning is less a matter of logic than of experience—and experience is usually hopelessly illogical. The benefits we derive from experience are determined by its logical analysis and ordering, but there must first be some experience upon which logic may operate and in relation to which it may develop.

The majority of mathematics text-books are admirable adjuncts to class-teaching, but they are quite unsuitable as means of exploring the subject in the hands of the pupil. The text is scrappy and appears to have been written on the assumption that it will not be read, and I think it no exaggeration to affirm that the main use of the book is as a store of examples, and in these circumstances it is impossible for the student to appreciate the fact that the subject may be read.

The " interest pockets " of the assignment afford some scope for remedying this, but it is all too inadequate, for the high lights are comparatively infrequent, and the work assumes the appearance of a collection of detached items of interest and of special tricks with a limited range of application.

My own attack on this problem took the form of seeking some dynamic unity, an endeavour to reduce the collection of tricks to a comprehensive and comprehensible principle of the widest generality.

Thus, in Arithmetic I seize upon what I have called the principle of Related Change. I neither use nor allow the use of the term proportion or unitary method, which is meaningless to the young pupil, who does, however, appreciate the meaning of change, and has plenty of experience of related changes. This, with the elaboration of a convenient thought system, covers the applications of arithmetic, and there are no longer different kinds of sums, each demanding its new method. In every case it is now necessary to appreciate clearly the quantities involved and how they are related, for he has a thought process which he can apply to such conditions.

Similarly in Algebra we claim to be fairly skilful in solving Simple Equations only, and all the supposed tricks of method for solving Simultaneous and Quadratic Equations are revealed simply as their intelligent management in an endeavour to bring these to the level of a genuine power, *i.e.* to derive from them Simple Equations with which we know how to deal effectively.

Following up this idea of dynamic unity through comprehensive principles I drew up some revision notes on fairly large manageable sections of the subject. My idea was that, in taking a single sheet of the notes, the pupil had all the essentials of that topic within his grasp, so that his previous experience might be compactly consolidated into effective knowledge. In this respect they served a useful purpose, but the exigencies of teaching problems (particularly at the Technical College) led me to adapt them to the requirements of the acquisition of knowledge as well as its consolidation. This extension of their use showed them to possess a number of advantages, not least among them being a distinct accession of confidence in the pupil, who saw a well-rounded job before him, the accomplishment of which marked a distinct stage in his progress. With the text-book extending interminably into the future it is not remarkable that the pupil feels overwhelmed by what lies ahead. With a single sheet he has a clear objective: he can go clean through the hoop. As successive sheets are mastered, he builds up his own text-book, and they remain a tangible memorial of what he has done, not an ominous foreboding of all that he has to do.

I know to my cost the labour involved in such an undertaking, and publishers will not risk the production of a book on such unconventional lines, and the problem of the text-book still remains a great difficulty for those of us who desire to develop our pupils' powers through their effective exercise.

Other difficulties of organisation and discipline will have suggested themselves to you, but I must press on to an examination of the principles of the Plan, and investigate the success with which they are realised in practice.

First, then, the freedom for which the Plan stands. The pupil is to be free to conduct his own management of the allotted task: he works as and when he will, and moves freely upon his lawful occasions within and among the laboratories.

I have been told that, human nature being what it is, such liberty is a mistake. My reply is that that attitude is based, not upon human nature as it is, but upon a travesty of human nature which we have made. What we pass judgment upon is a response to the environment: the nature of the individual remains for ever inaccessible to us, and our problem will not be solved by a pious belief in a state in which that nature will be perfected, but rather by securing the environment to which responses may reveal more of the essence of that nature.

Then, so far as it goes, this represents a great advance. It is regrettable, but has to be admitted, that we glory in the uniform progress of our classes, and we strive and drive to make them march together. It is regrettable, because there can be no more patent instance of self-deception than this, for the class that moves uniformly has but advanced in the successful imitation of certain tricks: its members have not grown in mathematical grace. Then we do but ill-service to our subject by such a practice, whatever pretended success may come to the Department in examination results. But to our profession we do a yet greater disservice, for such wholesale dragooning is the very antithesis of education. An army may be drilled by such methods, but the success will be commensurate with the extent to which the units will abandon their souls to the claims of mechanical precision: never by such means can we produce men. In these days of large-scale production, standardisation and mass-methods are the industrial rule and give the desired external precision and perfection, but only because there is no question of a soul to be considered. The work of your artist craftsman cannot be so produced, however faithfully you copy its externals: you can never implant that subtle spiritual quality which arose from his being himself. Now, examinations or no examinations, our work is essentially spiritual, and, however much varnish we put on our pupils by credits or distinctions, we do but decorate a shell by such methods. If as the realisation of themselves these things come, well and good, but if *we* polish and furbish them to such ends we turn out empty shams with certificates which rightly belong to us.

Then, if we regard our work aright, we must allow the pupil to be himself and not some travesty of it which we prescribe. This does not prevent us from elevating that conception of the self, revealing new and hidden possibilities and aiding in their achievement, but it does forbid us to fashion a mould of our own conceit into which to ram the soul-stuff in our charge.

We must therefore acknowledge the merit of the Dalton Plan in indicating the need for and the possibility of according such liberty to the pupil. Yet I am compelled on this very point to criticise the Plan, for I regard this proffered freedom as nothing but a sham. What is the position? In the place of oral exposition in class-teaching and the allocation of exercises for preparation, we have tasks prescribed on the printed assignment. The tyranny of the class-teacher has but given place to the tyranny of the assignment: the instructor still prescribes the task, and to do this through an assignment and call the result liberty can only be described as specious fraud.

That the student assumes the responsibility and contracts to do a job is an advance, but only because traditional practice refuses to give him a forward glance at his task. The very fact of his entry into school is an act of entering into a contract to do a job, but the luckless wight little recks what that job may be, and we lead him on to it blindfold. Making it a deliberate conscious act gives the contract more significance and may react beneficially on the work, but in actual fact I see but little difference to the contract assumed by entry to the school. The great gain lies in clearly envisaging a task of some magnitude.

But this itself invites criticism. The task is arbitrarily prescribed by the instructor. The monthly spread of work, the weekly allotment, the decision as to what constitutes a work unit, are matters for his judgment. Errors occur, small blame to him, and fortunately the mistakes are not generally fatal; usually they are not grave enough to be apparent, and over the whole they largely cancel out.

It is an accepted principle that the assignment is open to modification, and this affords a loophole in the case of graver errors: they can be adjusted in the next assignment. But this is again a matter for the judgment of the instructor, and so the merry game proceeds.

The fact is that as soon as you begin to interfere in the business of the living of another person's life you get into trouble: that is why teachers and politicians are such a nuisance. But traditional methods of teaching are based upon the right and ability of teachers so to interfere, and the practice of the Dalton Plan has not escaped this difficulty.

It is frequently said that the organised study methods of the Plan result in grave loss of inspiration coming from the teacher. There is a measure of truth in this, but the loss is not so serious as might appear. We must recognise that there are brilliant men whose lives are a veritable source of inspiration, and the class is fortunate which can benefit by the intermittent flashes of their genius. I say intermittent, for no man can consistently maintain this high level. But I hope that I have made it clear that collective work is not wholly abandoned: there is ample scope for the personality of the teacher to operate in the periodic conferences. Even so I would suggest that an aspiration, arising out of a high purpose from within, is much to be preferred to a stimulus coming from without. Moreover, much that passes for inspiration is just good, ordinary, honest perspiration, and its tragic weakness is too often revealed in the flabby lives lived after its withdrawal. The supreme need of our age is the aspiration of the creative artist, the poet, the musician, and I hold the scientist to be one with them: a mainspring of intelligent, purposeful striving arising from an awakened consciousness of a self that can flourish, an awakening made possible only by the opportunity for the self to break through its husk in an appropriate milieu.

Time allows me to raise but one further criticism, but its vital importance brings me to the crux of my experience: it marks the parting of the ways where I was brought to realise that I must cast about for new principles and evolve a new technique.

Miss Parkhurst says: "The second principle of the Dalton Plan is co-operation or, as I prefer to call it, the interaction of group life. . . . Conditions are created by the Dalton Laboratory Plan in which the pupil, in order to enjoy them, involuntarily functions as a member of a social community." I have to state frankly that my experience has not shown that to be true. Laudable as the intention may be, it has not been realised in our practice, and existing conditions make its realisation impossible.

Consider a school of 500 boys, each supplied with a month's assignment in each of, say, seven or eight subjects. With this sheaf of assignments, and faced with the need to get his card marked up, the pupil is likely to be too engrossed with his own task to spare a thought for social conditions: with his month's task he stands in isolation, and wants neither to bother about nor to be bothered by his fellows.

I had long felt that great opportunities were being lost by the impossibility of group work, where pupils could share in a common task, each contributing his quota to the solution of a difficulty which had proved too great for their separate efforts. I had tried to encourage this in my own laboratory, and boys made attempts to take advantage of the opportunity, but it was quite obvious that material conditions were too hopelessly adverse. I was much exercised in my mind to find a means of securing this new desideratum—

social activity in the pursuit of common ends. The outcome of much thought was a memorandum, which I submitted to my Head, the gist of which I give here.

The rigid disposition of desks is suggestive of servitude, and gives most of the boys eight immediate neighbours, and offers no reasonable assurance of privacy for undisturbed study. Groups for co-operative work cannot form under such conditions. The usual locker desk affords most inconvenient storage for books and materials, which are generally inaccessible, so that a habit of effective order is discouraged. The master's desk at the front of the class has long been a centre of doubtful associations, and boys do not readily walk out under the gaze of their fellows to consult him. The whole furnishing of the room is designed for convenient haranguing of the mass, and admits of no other use. These facts and traditional associations conspire to render the conditions of working undignified, and it is not surprising that adolescents are chary of being caught at work by their fellows under conditions redolent of all that is worst among the survivals of last century's educational practices—conditions which rob the actual work of dignity and manly desirability.

Admitting the force of my criticisms my Head encouraged me to remedy the evil, and placed at my disposal a number of boys, who made thirty plain lockers in the school workshop. These were fixed round the walls of the room, their vertical fronts falling to provide a writing surface. The rows and ranks of desks hitherto used were scrapped and replaced by three trestle tables in the now clear centre of the room, which thus came to present the appearance of a really decent intellectual workshop, and quickly displayed a number of advantages.

It will be clear that a room so equipped affords conditions which are much more elastic, and it is therefore better able to meet the widened and ever-changing needs of the occupants. The central tables give adequate and convenient accommodation for groups wishing to work co-operatively, while each boy can work in undisturbed privacy at his locker with all his books immediately accessible before him. Free movement is possible without causing disturbance or attracting attention, while the master's desk ceases to dominate the situation and becomes a centre of co-operative activity. The folding chairs can be easily transported for work at the centre, while the tables can be quickly collapsed to clear the centre of the room for work which requires a considerable amount of space. The tables also allow practical work to proceed quite conveniently. For occasional collective instruction the chairs are disposed in the most convenient manner, boys remaining at their places or sitting at the central tables as may be desired.

These were very real advantages, and working over a few months demonstrated unmistakably that I had moved in the right direction. But I must now skip a year to come to the final form of the equipment, for it was felt that yet greater elasticity would result from having each locker a self-contained, movable unit. I prepared several designs, and finally made a model which met all our requirements. Again I have to make my grateful acknowledgments to the Head Master, whose confidence and practical support in providing the sinews of war encouraged me throughout the experiment. Thirty of these new bureaux were made by the boys in the workshop, and they proved such a striking success in the course of the next year that, on the opening of new premises the Governors decided to instal this new equipment. Thanks to the courtesy of the Bennett Furnishing Company who supplied these bureaux, I am able to exhibit an actual specimen.

Meanwhile one of my colleagues had utilised my earlier lockers to investigate the possibilities of the practice in History, French, and Latin, and extraordinarily gratifying results accrued from this extension.

I must now revert to the first year of the experiment in which those crude lockers were used, for they merely represent the setting of the stage on which

the drama was to unfold itself. I cannot pause to speak of the inestimable social advantages arising from Form activities, indoor games, the supply of periodicals, pursuit of hobbies, social gatherings and concerts, but must press on to the more technical aspects of our work in mathematics.

At that time I was responsible among other things for the presentation of Form IVA for the General School Examination of London University in mathematics. Much of the ground had been covered, and the work consisted of its extension, consolidation and practice. Matters were still proceeding in accordance with our modified Dalton practice, and boys were working to assignments. My usual procedure was based upon the fact that some new work had to be done, and earlier work had to be consolidated and practised. The general plan of the assignment was therefore new work, exercises thereon, and exercises on past work.

It was this revision which first aroused doubts as to the wisdom of my ways, for, holding to my faith in the need to recognise individual differences, I saw that my revision prescriptions might serve the needs of some, while those of others were quite unmet. Careful observation soon showed that I was wasting a large percentage of the time of a large percentage of the boys. It is no answer to say that in the course of time all needs will be met, for this overlooks the fact that it is the incidence of a need and opportunity which is the key to effective and therefore truly economical effort.

I was thus faced with the problem of meeting individual needs in revision, and the only solution appeared to be to require each pupil to reflect upon his own standing and experience, and to assess and cater for his own needs. How to arrange for this and avoid chaos and loss of contact gave me much anxious thought, until I decided to abandon full assignments and replace them by skeleton assignments. These contained the bare essentials of new work and exercises thereon, which the pupil was required to elaborate into an adequate assignment for himself, incorporating such revision as he needed. Only a few carbon copies of these skeleton assignments were prepared, and the boys had to study those and construct their own detailed assignments on a sheet of paper, which they retained as their month's programme.

This may appear to be an unprofitable consumption of the pupil's time, but actually it offered a solution to a long-standing problem. In the Dalton assignment the pupil is faced with the task of completing his work units, and in mathematics the solution of exercises bulks largely in these. The result is that he tends to cut the cackle of my cleverly constructed " interest pockets " and goes direct to the exercises, with no gain to his understanding. Thus the need to elaborate my bare suggestions did at least forcibly direct his attention to a most important aspect of his task.

Although the pupils' assignments were submitted to me, I still had no means of keeping my finger upon the pulse of each boy's activity, and for this end I used the simple device of a card index.

On his draft assignment the boy was required to specify the nature of the new work and to give details of the nature and scope of his own revision. This summary was followed by the weekly allotment of exercises, including mine on the new work and those selected by himself for revision. When this draft was submitted to me, I had an opportunity of getting to grips intimately with the boy and his peculiar troubles, and a judicious word could be dropped to bear fruit in his subsequent work. The draft was then initialled as approved, and he was required to make me a copy on a card. On the front appeared the detailed weekly allotment of exercises, and on the back a summary of the nature and scope of his new work and revision.

Again I must emphasise the fact that, far from being a waste of time, this compelled a clear appreciation of the task both as a whole and as related in its details, a result which had never been truly achieved under the Dalton assignment.

So we worked in that Form, and the access of power and purpose and the unfolding of a new spirit were so strikingly beyond my expectations that I realised that this was no *ad hoc* device suitable only for an examination Form.

I suppose we all agree that at every stage of the learning process consolidation of past work must proceed in conjunction with new developments. That was certainly my attitude, and I felt the impossibility of meeting all needs in revision no less in the earlier than in the later stages. I was therefore committed to the investigation of the problem with my lower Forms, and it was in this extension that I received the shock which brought about the collapse of the whole structure and compelled me to build anew on better founded principles.

In one of my skeleton assignments for Form IIA, average age twelve, I gave the instruction, "Compile a list of formulae for Areas and Volumes, and know how to use them." They were familiar with their development in the simpler cases, and I was content at this stage that they should have a complete set of tools. To my astonishment practically every boy began to interest himself in their *development*, and brought along difficulties encountered in various books consulted. The final blow was delivered when one small youth, generally recognised as not over willing or able, asked how to develop the formula for the area of a sphere—a matter carefully avoided in all his books. Like the irritating pedagogue I was, I told him that that could not be done yet for want of knowledge of the cosine of an angle. He was momentarily abashed, but quickly recovered himself and gave me my just reward by asking, "Can't I learn that now, Sir ?" All my conceits collapsed around me, and I realised that I was no sounder judge of the rate and order of progress in new work than I had previously admitted myself to be in respect of revision.

Fortunately I did not consider these incidents as evidence of my own incompetence, but as residual phenomena in a series of experiments whose further investigation would carry me nearer to my goal. But the rashness of the next step called for all my faith in the principles. However, I realised that the skeleton assignment must go, and that the pupil must be allowed to see the new country he was to prospect, and to choose his own routes through it. Thenceforward he was made responsible for the entire designing of his assignment, and he now prepares his draft, for a month if he wishes, or a fortnight, or even a week, submits it to me, and finally makes me a copy for my card index.

On these lines the work proceeded smoothly and satisfactorily, and feeling my way cautiously I had even extended it to our first Form without encountering difficulties. But for some time I had felt that mental work was being neglected, and I was loath to provide for this on a collective basis. A chance discussion with my colleague on oral work in French suggested a method on the lines of Norman MacMunn's differential partnerships. Instead of preparing the questions myself, however, I decided that each boy should compile his own set of five or ten questions and submit them to me with their answers. After approval, he attacked a sparring partner in return for the privilege of being set upon by him at the same time. Each wrote the answers to the other's questions, and these were exchanged and checked and the results recorded on a Form list supplied to each boy. At convenient times the results were brought to me, and the scores for and against were recorded on the back of each boy's card.

Following that historical sketch of the growth of the method, let us examine it in practice with a typical Form in mathematics.

It is the beginning of term, or half-yearly examinations are over, or it is any other convenient time. For a week or so we shall be engaged upon informal discussions of a very broad nature. In Algebra we may talk of functionality, formulae, direct and inverse variation, formula manipulation,

evaluation and graphical representation. That may stand as typical of discussions in different branches of the subject. It is no one-sided affair: the boys participate freely, raising doubts or opening up a line of thought that has been suggested. Over all the subjects a term or a half-year's work may have been outlined, although with younger boys less will be attempted and such conferences will occur more frequently. We stand on a Pisgah height and survey the promised land: unlike Moses, I have explored it, but I am taking no conducted tours through it. Each must explore it for himself, prospecting for its treasures, though as an old-timer I am at hand to help or advise our tenderfeet. My function is to provide the opportunity, with all that this may mean, and I am no longer described as teacher or instructor, but as provisor.

Following this panoramic view I retire to the unobtrusive seclusion of my desk. Boys will spend an hour or two in planning their work for a whole month or less as suits their capacity or inclination. Each has his own programme, cards are filed in my index and the work is under weigh. Boys are busy with their tasks in undisturbed privacy at their bureaux or in collaboration at a central table. Difficulties arise and are surmounted, or they are brought to the provisor for help. All branches of the work are proceeding simultaneously: there is no reason why practical and mental work shall not be in progress at the same time as the study of new matter and the solution of exercises.

Weekly allotments of exercises are done and marked by the boys. If the work is satisfactory, it is placed in the appropriate compartment in a rack at the side of my desk: if error has arisen, the exercise still affords a field of valuable research for the boy. He must find his error, ring it round in pencil and briefly indicate its nature. If it eludes detection, I am consulted here as on any other difficulty.

Betimes I have the opportunity to examine the work submitted, being watchful in correct answers that I am not being bluffed, and that the pupil is not deceiving himself by a correct result arising from mutually cancelling errors or any other cause, and in the case of detected errors satisfying myself that the pupil has gone to the root of the trouble. Approving of the work as a worthy effort and likely to serve the boy well in his quest, I mark it to indicate the fact, and run a red ink line through that section of the work on the face of the card.

As assignments are completed, the card is dated and filed at the back of the cabinet, and a new one commenced. In this way the boy's progress can be watched through each assignment and also throughout the year. As the months pass, reference to earlier cards will suggest new matter to be incorporated in future assignments, or points of contact between new and earlier work.

At appointed times different Forms enter the room; no instructions are given, and the boys settle down at their places and proceed steadily with their work.

Occasionally a deputation will wait upon me with a request to deal with a more general difficulty. If it concerns a small group, it may be discussed at my desk or at a central table, but if it extends to the whole Form, we appoint an occasion on which to examine it. These are a constant source of wonder to me, differing as they do from the usual lesson in the intensity of the eagerness with which the boys hang upon one's words, because these are dynamically related to a felt need.

In view of the fact that I give no lessons and set no work, I am often asked how the slacker fares. I will not pretend that I was not anxious on this score, but I decided that I might justifiably take some risks in order to discover exactly how he did react. This is not the occasion to discuss fitness for a secondary education or the mode of selection of those who shall enjoy it. There doubtless are pupils in our schools who should not be there, but I am

concerned with that range extending from the eager boy down to the individual who will dodge your work if he can. And that is the crux of the matter—he will dodge *your* work. So long as he has the idea that he is doing your work, he will feel that it is part of your business to get that work, and he will take no steps to help you on that score. I have seen it now for a number of years: I have seen many determined attempts to eradicate the evil with an exhibition of "pull devil, pull tailor." I am not saying which party wins: the work is done, of course, but I ask what is the worth of such work. In the early stages youths of that temperament showed their doubts of my sincerity, but I quietly maintained my attitude without comment, and they found little satisfaction in a tug of war in which only devil or tailor pulled, landing on his back with a length of slack rope in his hand for a reward. I claim no brilliant reform of character, but certainly a lesson was learned, and my determination was appreciated as a far greater reality than ever I had found it to be when shown by my insistence upon my demands.

When the pupil realises unmistakably that it is *his* work, that it is his very self that is concerned, he is slow to advertise his crass stupidity in the usual manner. There can be no doubt that this redistribution of emphasis is an urgent necessity, and I have found it secured by my practice, where the work is essentially the boy's own, not merely in its doing as in the Dalton Plan, but in its very inception, conception and purpose.

The difficulty of the spread of the work arising from the arbitrary prescription of the instructor has also disappeared. It is now the affair of the pupil and is related to his capacity and needs, and is open to adjustment according to these. Boys make errors in designing assignments, but so did I. The great gain is that the boy appreciates the fact and meets the situation intelligently. It is no uncommon experience to be approached with the suggestion that this particular question proves to be very easy and not worth the doing, or that this point will be more effectively attacked after doing another piece of work. In such cases a more profitable exercise or piece of work is substituted, to the great advantage of the work itself and with incalculable benefit to the boy in managing his experience.

The lack of class-teaching criticised under the Dalton Plan is not an objection with which this practice can be charged. The matter was raised by one official visitor, and I replied, pointing to one or two groups working together, "Where two or three are gathered together . . . ," and he saw the light. But there is more in it than this, for in such groups the teaching function is constantly shifting and is in the best hands at every moment, and the boys get those undoubted advantages accruing from the difficulties of exposition and adjustment to another's point of view.

I have been able to say nothing of differentiated curricula, the time-table or promotion, and I must now very briefly pass to one or two important emergent principles.

While I stand firmly for the claims of the individual, I wish to make clear what the individual signifies to me, for it is upon this rock that all systems with which I am acquainted are doomed to split. Their protagonists have realised the existence of great individual differences, and have elaborated schemes to meet them. But individual differences do not make the individual: to me the differences are incidental and subservient to the needs of the individual. I stand for a unique concept which I call individuality, not for a medley of individual differences—for the individual, the not divided.

Now this individual is not the unique entity that he is in isolation, but by virtue of his being in relation. Superficial biological views do a great disservice here by leading to the idea that the creature passively submits to the conditions of the environment and is modified thereby. I would go so far as to suggest that the environment is not given in this simple manner, but has to be achieved: it is not by the relations in which one finds one's self that

one is fashioned, but by those relations into which one effectually enters. I therefore regard the individual as an act and a fact of achievement, an achievement of his environment, an achievement of his self. That is the basic philosophical principle of my method: the active discovery, the actual achievement of the self, and I regard Mathematics as I regard any other subject or phase of experience as instrumental to this end. But chief among them, indeed, permeating them all, must be the social environment. The boy must live with his fellows, and I therefore urge the importance of provision for co-operative activity in the pursuit of common interests. In any or all of these fields our pupils are prospecting for self: what the lode will be—silver, lead, or gold—we cannot say, but at least let it be good metal without flaws, and above all free from that abominable imposture of the present age, a hall-mark not appropriate to it. G. W. Spriggs.

[Enquiries suggested by this paper and discussion, and information as to cost of equipment, etc., may be obtained from Mr. Spriggs if addressed to Tiffin Boys' School, Kingston-on-Thames.]

Discussion.

The **President** said that the Association was very grateful to Mr. Spriggs for his discourse. He did not know whether stimulation was to be deprecated after what Mr. Spriggs had said, but, if not, then this paper was certainly stimulating. The Association suffered to some extent from the lack of any common aim, but all the members were extremely interested in anything which gave better play to the individuality of the children with whom they had to deal, and he was quite sure that there were many present who would take what Mr. Spriggs had said very much to heart. He hoped that there might be some who would definitely experiment on these lines, which were new, and to some might seem extreme. Teachers valued highly the stimulus—to use the word again—that could be conveyed by class work, and it must be a little difficult to throw over previous conceptions and start on such a system as Mr. Spriggs had described. But the results seemed to show that it was worth doing.

One or two points came into the speaker's mind, during the reading of the paper, on which perhaps some comment might be made. When Mr. Spriggs was replying at the end, he would like him to say, quite definitely, what his views were on the question of error in a pupil's work. If the pupil had made a mistake, but could not see where the mistake was, was he to be compelled to go on and get that particular bit of work right, or was he to be allowed a little latitude? Another difficulty was with regard to the pupil planning out his own work in the way Mr. Spriggs had proposed. It hardly seemed as if the pupil could do this effectively unless he had some sort of general view of what that work comprised. If he had only the work that he had done to go upon, he could not plan out anything new. How was he to get, not so much a knowledge of the new, as a knowledge of what the new consisted of, so as to enable him to make this plan effectively?

Mr. **Hope-Jones** asked what happened when, owing to illness, there was an interruption of, perhaps, two or three weeks' work. He wondered whether the work was more likely to be cut up as a result of absence if it was done on the Dalton system than if it was done on the old-fashioned system. He also wished to know whether Mr. Spriggs found that the use of technical terms was called in excessively early by this plan of envisaging the work ahead. Mr. Spriggs had mentioned "functionality." Most teachers preferred to accustom the boy to functionality by concrete instances, and such things as graphs, before introducing the name itself. His own plan was to avoid the technical name of a subject until the boy had done something on it, and to

tell him the name afterwards when he was in a state of mind to be less terrified by it. It seemed to him that there was a definite advantage in the teacher himself choosing the questions, because he chose questions which he knew from his own experience illustrated well the principles involved. This advantage, he thought, might largely be lost if the boys themselves chose the questions. As a Parthian shot, he recalled that three or four years ago Mr. Fletcher gave the Association a paper on the Relation of the Teacher of Mathematics to good English, and he would like to suggest that the paper which Mr. Spriggs had just given was an excellent illustration of the value of good English to the teacher of mathematics. He had never heard a paper given before the Association which he had more enjoyed, not only because of the substance of it, but because of the phraseology. (Hear, hear.)

Prof. **Piaggio** (Nottingham) asked whether he had understood Mr. Spriggs to say that he did not use text-books but note-sheets. Were these duplicated or printed? He also wished to say that a parent, whose boy was being brought up on the Dalton plan, told him that he looked at his exercise book and saw that a large number of sums, which were wrong, were not marked as wrong. If the boy was left to mark them for himself, was there not a danger of the wrong sums being overlooked?

Prof. **W. M. Roberts** asked that when Mr. Spriggs replied he might settle a point which would otherwise cause a little heart-searching. Mr. Spriggs had rather given the impression that the teacher under the Dalton plan did not himself do very much work. He thought it would be well to dispose of that idea.

Dr. **Jessie White** asked whether the boys had been accustomed to free work before they came to Mr. Spriggs, or were they brought up under the ordinary regime?

Miss **Mielziner** (Liverpool) asked whether Mr. Spriggs had any information as to the number of schools in England which were carrying out (1) the full Dalton plan, (2) the modified Dalton plan, and (3) occasional experiments in the Dalton plan. It was very refreshing to hear of new methods being carried out in schools. At a meeting in Liverpool of the New Education Fellowship, the head mistress of a school which carried out the modified Dalton plan was present, and it came out at that meeting that the chief difficulties in the Dalton plan were in French and mathematics, so it occurred to the speaker that it would be very valuable if the Association made some investigation of this subject and gave statistics as to the number of schools carrying out these different forms of instruction. It might then be possible to go further and study other methods of individual work from America, Belgium and Russia.

Mr. **J. Katz** said that he saw some of the working of the Dalton plan at the Streatham County School, and he talked to some of the girls there about their algebra. They told him that they found the text-book shockingly dull, and they felt that they wanted very much the assistance of the teacher, but that there were so many pupils who wanted the assistance of the teacher at the same time that they did not get anything like the show which they were able to get under the older system. He would not put that forward, however, as a serious criticism. His own experience of the working of the Dalton plan was that it was very largely a calculated and valuable system of make-believe. The pupil thought he was really originating something, whereas, as a matter of fact, it was the teacher who was making the suggestions. Those initial conferences about which the lecturer had spoken in which the teacher gave a sort of general view of the possibilities of the subject were, he rather fancied, crucial in the system. It was there that the child got the suggestions as to the possibilities of the subject, and he was probably not aware how many of those suggestions came from the teacher, and imagined that he was himself a great originator. It was very valuable that he should have that illusion,

and a system which induced the child to think of himself as an originator had merits. But the teacher himself ought not to be under any illusion that the child was a marvellous originator. As for the teacher's work, he thought that probably, under the Dalton system, he would have to work five times as hard as he did under the present system. Instead of the teacher lapsing into idleness, as had rather been suggested, the teacher was called upon to do much harder work.

Miss **Belle Rennie** (Hon. Secretary, Dalton Association) desired to say how much she had appreciated the paper. She thought the Association had been listening to a real pioneer in education. She felt that there was a danger of the idea gaining currency that lessons were swept away. Those responsible for the Dalton plan had always striven to strike the right balance between teaching and learning. They had said that the child should be left for a good proportion of his time to do his own learning and his own research. The question had been raised as to the number of schools working on this plan. She was always being asked to find out these numbers, and she had replied that if she were given plenty of money and a staff of clerks she would obtain the information, but her time was occupied by sending out literature to enquirers and by arranging for visitors from abroad to see the schools here. But there were many more schools working on the Dalton plan than was commonly supposed. For example, in Woolwich she knew of one Dalton school—not entirely Dalton, but Dalton in the top classes—and she asked for some others, and was given the names of eight others in Woolwich alone.

Mr. **Wright** (Winchester) said that it seemed to him that the choice of the work to be done by the pupil had got to be regulated by something, and it could not be discovered by the pupil himself. Mr. Spriggs had referred to the growth of a seed, and the necessity for rain and sunshine. But what if the seed happened to be that of a sweet-pea, which had to have a stick on which to climb ? If the pupil was going to choose his assignment, the sweet-pea was choosing its own stick. He had to find the stick from somewhere, and if he did not find it from the teacher he found it from a text-book. Further, was the pupil expected to look ahead from one job to another ? The able child every time would be mapping out his own work from the book, but he would not be able to invent it for himself. Given the book, he could go ahead ; but the less able pupil had to be guided, and it was necessary to explain it to him. He had not yet found a book that the average boy could really read and assimilate by himself. The ordinary teacher used the ordinary text-books and made his own exposition supplementary.

Miss **Tiverton** (Ipswich) said that she could visualise a boy very keen in mathematics getting far on in his work on that subject, perhaps to the neglect of languages. What would happen to that boy later on ?

Mr. **Taylor** (Croydon) asked how Mr. Spriggs dealt with such a subject as mechanics, which had allotted to it a very little time.

Miss **Brown** (Liverpool) said that if a boy made an assignment for a month, it was very likely that after a certain time he would want to alter it. Was any opportunity made for modification of the plan ?

Miss **Thomasson** asked if the pupils of whom Mr. Spriggs had spoken had to sit for examinations as conducted at present, and how they compared with boys and girls brought up on other methods.

Mr. **G. W. Spriggs,** in reply, said : With regard to the marking of exercises and the treatment of errors, I do not require a wrong exercise to be done again and put right. What I desire is that the boy shall discover his weakness in relation to that exercise. If he simply repeats the exercise, and by a stroke of luck gets it right, he will trust to the same luck to serve him on the next occasion ; but if he discovers the real reason for his failure, then he has something definite which will serve him in the future.

The President's second question has already been well answered. There are differences which we must always bear in mind between the way in which we ourselves recognise things and the way in which our pupils recognise the same things. We are looking at them from different aspects, and they have different meanings for us. Therefore, although I recognise the vital importance of the bird's-eye view—the Pisgah view—and keep it constantly before my mind, I do not let the boy see all the cards in my hand. I have to watch that these occasions serve his purpose, and I have to regard them very carefully from that point of view. I do not say that he knows much about it by the time we have finished, but he must, at least, have been very keenly stimulated. It is only in so far as these things arouse an interest in the future work that they serve any useful purpose at all, and if this cannot be done in the course of the preliminary broad view, one had better find something out of the book.

Metaphors are clearly dangerous implements, and my reference to plants cannot be pressed too far. If a sweet-pea must have a stick, then one must be provided: that is the main function of the Provisor—to provide the opportunity and all that that implies. But I fear the danger of unduly weedy growth encouraged by injudicious measures which thwart the development of well balanced powers in the endeavour to produce attractive monstrosities in the form of prize blooms, and by an over-willingness to supply a prop which may discourage the development of native power.

Then the question of the use of technical terms was raised. I bore in mind the fact that, in the course of my remarks this morning, I was addressing the members of the Mathematical Association and not the boys in my Form, and therefore I did not hesitate to resort to technical terms. I merely wanted to bring home the principles to you as briefly as I could, and that is one of the advantages of technical nomenclature. But I dare not say to you some of the things that I say to my boys. I have certain ideas to get "across the footlights" to them, and language, after all, is only an instrument. If I can get my point home better by taking liberties with the language then I do so. Of course I would not use those technical terms in the way that might have appeared from my remarks. In the preliminary discussions the need for such terms will rarely arise. I talk colloquially to the boys, and I leave them to get the shocks with regard to the terms when they come to the text-book. Then a boy will probably ask me what a particular phrase means, and I tell him that it means just what I meant when I said so-and-so. He then often asks me why the text-book itself does not put the matter in the same way.

The question of selecting exercises is a serious difficulty if one views it from the standpoint of the customary mode of conducting a class. The teacher is undoubtedly the best person to select a large number of exercises for a large number of pupils; but the position is entirely different when one comes to consider the meeting of the requirements of the individual. The "hit or miss" method of which I am rather acutely conscious for selecting a bunch of exercises to be done for preparation luckily does not involve any serious errors. The errors largely cancel out, as I have said; but the method does fail in one important particular—it fails to secure what I may call the incidence of a need and an opportunity. I believe a great deal of importance attaches to that. I will not go so far as to make the charge that the master chooses the exercises which are convenient for himself, but I do raise the question as to whether he sets exercises which are vital to the pupil. Does he set the questions which the pupil will find purposeful? There is a further remark to be made upon this matter of the difficulty of selecting satisfactory questions. I admit that there are important problems in the method of selection I have put forward, and that the boys will make mistakes, but I do not consider that an error in that respect is altogether a disadvantage. Anyhow, I would

rather the pupil profited by his own mistakes than by mine. I do not stand up and say, "Do as I say, and not as I do." I would rather that the pupil made his own mistakes and profited by them.

In reply to another question, my revision sheets are actually duplicated, but please understand that these were intended primarily as revision sheets to consolidate the work, and they are a very makeshift device. Certain of the schools with which I am familiar and which are trying to work on these lines find the same acute difficulty in the matter of text-books, and I have ventured to meet this need in text-books written from the student's point of view, to make the text important and the exercises subservient to the needs of the text; in other words, to make the exercises open up the various aspects of the text. But I do want the student to have a text to which he can refer and is pleased to refer. It is very difficult to strike the happy mean between being strictly logical or truly academic and being almost vulgarly popular, but I am going to try it out on my own pupils.

In geometry I do use my own notes, which give a sequence of propositions, rather than use a text-book. A few years ago we did use text-books, from which the boys got a great deal of fun in criticising the authors. The authors would say things which "set their backs up," and they criticised with some freedom, and managed to learn the art of geometry in that way. The boys were extremely keen on such methods of criticism.

The marking of geometry riders is certainly in a different category. There are certain fundamental facts of wide application in the case of geometry riders. If congruence is involved as a necessary step, I want to be quite sure that the essential facts are there. That I leave to the boy. As you can appreciate, there will be several other very particular aspects of geometry in which we can say to the boy, "Now, there you have to watch *these* things." If any one of these aspects crops up, I want to know that the boy is investigating any that he sees to be defective, and then I am satisfied; he need not go through the mechanical process again. It is sufficient for him to say in what particular it is defective.

Let me, please, remove one misunderstanding. Every exercise is examined. I cannot understand how so important a service to the student could be evaded. That is certainly not the Dalton plan, nor anything else but an abuse of an opportunity. To save myself trouble and to save the boy's time I have, on the right-hand side of my desk, a rack with as many compartment as there are Forms to be taken, and on the other side a similar rack with similar compartments. The pupil places the work in the appropriate compartment on the one side; I mark it and place it on the other side, from which he takes it again. Thus there is no need for him to wait for me if I am busy on another task, though if any matter arose in connection with the marking of his work I should find a convenient opportunity of getting him alone and having a talk with him. But every piece of his work is marked, and, by the way, I always insist upon knowing the time the boys actually spend over the work, and if they desire it I give them some idea of the worth of the work.

With regard to the month's assignment, there is a troublesome idea at the back of one's mind that a month's work is somehow related to the movements of the moon and cannot be altered. There are about thirty-two working weeks in a year, and this gives eight monthly assignments. My boys plan their work for a month, but if, say, at the fifth month I find a fellow in the middle of assignment No. 4, and he says to me, "I am rather behind in my work," I reply, "How do you know that? If you have done five months' work over four monthly assignments it cannot be said that you are behind in your work at all." There is no virtue in the figure 5 over the figure 4. It is a question of the spirit and purpose behind the work. We want to get away from some of the fetters of elementary numerals. My contention is that even with the smaller amount of work that may be done in some cases, work

is so purposefully related to the child's needs that it is no less valuable, at any rate, than a great deal of the purposeless work previously required of him.

The question asked with regard to the number of schools using the Dalton plan has been disposed of, I am glad to say, and I may add that my pupils have had no previous experience of working on free methods—their earlier education has proceeded along traditional lines of mass methods. I ought to point out that the practice adopted at the Tiffin Boys' Secondary School is a departure from the Dalton plan in one respect. We work on a time-table with regard to the use of the laboratory, but whether I should do the same thing if I could find a philanthropist who would build the school I want, and would let me run it as I would like, I cannot say. I am not complaining of the time-table in the least. The balancing of studies, of course, is secured as soon as the time-table is in operation. As to mechanics, this can be handled practically in my laboratory. There are one or two admirable little books which give sufficient guidance in respect of practical work, and these may be followed with simple apparatus on the central tables.

The assignments are open to modification. I have brought a number of assignment cards, and in some cases it will be seen that the word "transferred" has been written across them. This indicates that the student has assessed his position and taken a line of action to meet it.

With regard to examination results, I suppose we do have to pay some attention to them, and I can only be fair here and state the facts. I worked on the modified Dalton plan with Form 4A at Tiffin's, and at the end of a four-years' course the Form did reasonably well, that is to say, we got a normal spread of results. There were one or two failures at one end and a couple of distinctions at the other. That was a reasonable success for the school and was in keeping with the general run of things, which suggests that under the Dalton plan no worse results ensued than under other methods of class teaching. The next Form I presented was on the lines of the free method of working which I have indicated. The number in this Form was about the same as in the Form just mentioned, namely, 26. Each year there were a couple of failures, but as to the results of this second year I will not give you my own opinion, I will give you the frequently expressed opinion of my colleagues. It was that the Form I presented on this occasion was quite moderate, it had a good head of about three boys, an unwieldy tail, but no body! The results of the examination were these: there were two failures, but all the rest reached matriculation standard in mathematics. Of the previous set, ten reached matriculation standard in mathematics, but in this second set all reached matriculation standard, and ten of them obtained distinctions. You will understand that I claim that for the boys, because they did it all. Between them they took up twenty-eight distinctions in all subjects, three of them got honours certificates, and my contention is that the achievement was very significantly their own.

A vote of thanks was accorded to Mr. Spriggs on the motion of the President.

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730. On the lecture slate
The circle rounded under female hands
With flawless demonstration.—

Tennyson, *The Princess*, ii. 493.

731. Le goût des Mathématiques, que j'ai été à portée de reconnoître en vous dans un âge moins avancé, et que vous avez conservé au milieu des travaux glorieux qui assurent à votre nom l'immortalité, m'a paru un titre suffisant pour vous en faire l'hommage.—Euler's *Introduction à l'Analyse Infinitésimal* (An. iv.-v.). From the dedication to *Bonaparte, Citoyen Général*, by the editor, J. B. Labey, Napoleon's mathematical teacher at the Paris Military School. [Cf. Sotheran's Catalogue 816, item 398, Dec. 1929.]

ARITHMETIC OF CITIZENSHIP.*

By B. L. Gimson, B.Sc.

In putting forward a plea for the inclusion of "Arithmetic of Citizenship" in the regular arithmetic course in secondary schools, there are two points which I want to make clear: (i) the purpose of such a course, and (ii) suggestions as to how it may be carried out in practice.

The first seven or eight years of a child's acquaintance with arithmetic are spent almost entirely on the mastery of the elementary processes of calculation: counting, learning the four rules, the multiplication table, extension of the fundamental operations to fractions and decimals, the methods of proportion, and so on. In the early years these processes by themselves are of sufficient interest to occupy the child's whole attention. For a long time he is satisfied by the mere appeal of numbers for their own sake. This is not to imply that the wise teacher does not introduce practical problems into the classroom. Indeed the whole business of weights and measures is learnt by actual weighing and measuring, and money sums are made real by simple examples in shopping.

But there comes a time when the growing boy and girl begins to wonder what it is all for, and the arithmetic text-books do not seem to be very helpful in answering this question. They offer him chapters on Percentage (Easy), Simple Interest, Compound Interest, Percentage (Harder), Approximations, Rates and Taxes, and finally (as the groans of dissatisfaction become still louder) Stocks and Shares.

Now I am not saying these are not worthy subjects of study, applications of arithmetic with which all need to be acquainted; but I think we go the wrong way about studying them. At this stage a child's flagging interest needs stimulating by fresh material upon which to try his skill in number manipulation. Many find this stimulus in the generalised processes of algebra, in the numerical applications of geometry, and perhaps best of all in the new field of numerical trigonometry. But there are still a great many children to whom this academic course fails to make its appeal. They are groping for something more human in its application, something that shall touch their lives personally; in fact, something in which the utility of arithmetic shall be immediately obvious.

For such children a course in Arithmetic of Citizenship is, I believe, the very thing they, and we, have been looking for. For the last five or six years I have been experimenting with a course of this kind, arranged for a class of boys and girls whose inclinations were decidedly unmathematical; yet their interest in the human aspects of the work has been sufficiently keen to carry them triumphantly through the necessary arithmetical drudgery.

The outline of the course has just been included as an appendix to the *Report on Elementary Mathematics in Girls' Schools*. Sufficient detail is given there to indicate the topics discussed, and hints as to general procedure. For the moment it will be sufficient, perhaps, if I recapitulate the main headings. Before doing so, however, I should like to be permitted a digression in the form of a mild protest against the course being considered as food only for girls. It was not so planned, and in practice the boys of my mixed classes seem to find it suits the male digestion equally well. The topics introduced affect the lives of men quite as much as women. In the problems selected for their attention no effort is made to appeal to girls rather than to boys. In fact the value of the course, such as it is, is not meant to lie in its fitness for a particular sex, but in its appeal to the child mind at a certain stage of its development, be it boy's or girl's.

* An Address delivered on 6th January, 1930, at the Annual Meeting.

The main headings of the syllabus are :

I. Local and County Finance.
II. National Finance.
III. Saving, Banking, Investing (from the investor's point of view).
IV. Capital and Industry.
V. Insurance.
VI. Compound Interest in Finance (discussed without reference to algebraic formulae, or the study of progressions).

Under these headings we bring in all the arithmetical topics discussed in the orthodox text-book ; but instead of focussing the attention on a particular arithmetical process and driving it home by applications drawn from various fields, we reverse the process. We centre the attention on a particular field of human interest, say National Finance, and we bring in the arithmetical processes to elucidate problems in that field.

For example, it is obvious that percentage must come in over and over again. It is the simplest way in which the figures of a budget can be compared, either among themselves or with those of past budgets. Unwieldy millions of pounds become intelligible even to school children. The reckoning of profits by percentage falls into its natural place when discussing the working of limited liability companies, and the dividing of profits among the different classes of shareholder.

So the application of percentage to interest arises inevitably when we take up the subject of saving and investment of capital. Then, too, the limitations of simple interest are made evident as perhaps they are not in the ordinary text-book ; and the student of insurance and of payments by instalment begins to realise the important part that compound interest plays in everyday finance.

It has always seemed to me a pity that this important mathematical idea of compound interest should be denied to so many humbler students of mathematics because they do not reach geometric progressions in the algebra course, or have but a feeble grasp of logarithms. In our Citizenship course we are able to tackle all the more difficult problems of annuities and repayment of loans by equal annual instalments, by the simple expedient of building up our own Compound Interest Tables, starting with the ordinary calculation of amounts by successive addition. From this we calculate a table of amounts to which a £1 annuity will accumulate, and finally a table of the present value of a £1 annuity paid for so many years. By the help of these three tables, calculated by simple arithmetic involving nothing harder than the division of a decimal, and made to appeal visually by graphs, there is no ordinary problem in present-day investment which cannot be solved.

It is worth noting that the subject of Stocks and Shares is met with twice in our course, first from the point of view of the person who has some capital to invest (and then we use newspaper quotations rather than a text-book), and secondly from the point of view of the formation of a company or the need of an existing concern to increase its capital.

Perhaps one of the strong points of the course is the natural way in which topics are repeated ; being viewed each time from a different angle, while enabling the teacher to consolidate with each repetition what had been learnt before.

In offering this syllabus for your discussion and criticism I should be sorry to put too much stress on the order in which topics are taken up with the class. The above is only one of many possible arrangements. It is not meant to follow a logical sequence, nor even to advance from the easy subjects to the harder. Personally I should let circumstances at the moment, the wishes of the class, or possibly the season of the year, play a large part in determining the order. For example, as April approaches I usually take the opportunity to talk about the Budget ; we may indeed have spent the better part of the

spring term discussing National Finance. It is encouraging to hear later of the interest with which members of the class have followed the Chancellor of the Exchequer's new Budget, discussing points with their fathers, sometimes to the latter's exceeding discomfiture!

Last term the Hatry case was an opportunity for a digression on the risks of investment. We distinguished trust securities from more attractive but less stable forms of investment. We discussed the part that credit plays in business. These digressions I welcome if they arise from a natural curiosity on the part of the class or bear closely on the happenings of the day. Incidents such as the gift to the nation by Lord Inchcape of £500,000 to be invested at compound interest for the purpose of reducing the National Debt are heaven-sent opportunities to the teacher of Citizenship Arithmetic.

The syllabus already outlined makes no mention of household accounts. We take for granted that pupils studying the Arithmetic of Citizenship already have some acquaintance with account keeping. This does not imply a knowledge of bookkeeping by double-entry, but at least a thorough grasp of ordinary cash accounts, together with a familiarity with the procedure of running a bank account. If possible boys and girls should be encouraged to keep their own accounts. Economy begins at home, and interest in a National Budget is made all the more vivid when one has tried to balance one's own expenditure against income, and perhaps budgeted for next year's spendings on the basis of past experience.

After a week or two of account keeping, which has included an exercise in the cost of feeding a family of four for a week, we always get our boys and girls to make out a budget for what they imagine it would cost a family in middle-class circumstances to live for a year. First we agree on the main items to be included in the budget: food, housekeeping, rent, rates, taxes, clothing, motor car (if any), education, amusements, insurance, and so forth. Then each child goes ahead and makes his guess at the supposed annual cost under each heading. They soon find themselves hopelessly at sea. "What exactly *are* rates?" they ask; and when they hear that the item depends indirectly on the rent of the house: "Yes, but I don't know how much you have to pay for rent." At first we do not give any help with these guesses. It is amusing and quite instructive to see the quaint efforts at a reasonable figure. One girl puts down £150 for dress; a boy enters £300 against education and £5 for gifts and charities. At length the total is arrived at, and in the majority of cases we find the figure exceeds £1000—and this for an average middle-class family of father, mother, and two children, or one child and a servant!

Then we begin to discuss things; we talk about average incomes for such a family, the kind of house they live in, whether they can afford a motor car (this alone may occupy our attention profitably for several lessons); and presently we get down to about £700. Under protest, however. Many refuse to believe that it can be done for less. If this feeling is acute, we take the opportunity to examine some household budgets of the very poor, such as you may find with every detail poignantly described in Mrs. Pember Reeves' *Round About a Pound a Week*.

One or two of the class complained that these examples, being taken from the pre-war years, were not a fair indication of present-day prices. This led us to discuss the Cost of Living index figures published monthly by the Board of Trade. Graphs were drawn, and we came to realise the fluctuations in the purchasing power of the pound. One boy grew so interested in this historical aspect of the matter that he spent three weeks last term in collecting data on the costs of various commodities and changes in wages since 1066, amassing a book full of some 500 items carefully classified.

Enough has been said to show the manner in which we approach the work of the course. We get our material as far as possible first hand. We have

no text-book to follow, but gather the data from Local Rate Demand Notes, Income Tax Forms, Company Prospectuses and Balance Sheets, Insurance Pamphlets; and for statistical information of recent events *Whitaker's Almanack* is a mine of information, which can be supplemented by the *Statistical Abstract for the United Kingdom* published by the Board of Trade.

Classes soon begin to feel that the examples set before them are live problems, for which there is no " answer " ready prepared at the end of a text-book, but an answer which requires their critical faculties and judgment to determine whether it is " right " or " wrong." In fact the problems are problems of everyday life, such as any householder may meet in his efforts to keep the family happy on an income which is always a little too small, or such as any reader may come across in his daily perusal of the newspaper.

In order to train the judgment in dealing with figures I think it is well to stress the need for constant comparison. Compare the figures for this year with those for last. Make use of historical summaries of statistics, and above all make use of graphs. Facts and figures by themselves are not of much value. Only when they can be compared with past figures, and estimates made of what they may be next year, do they take on an interest and meaning which may be both fruitful and suggestive of new problems.

One more point, perhaps, needs remark. The object of the course is not to acquire a knowledge of facts and figures in connection with the study of statistical data. There is no need to memorise the size of the National Debt or the scale of allowances in Income Tax rebatements. The figures themselves do not matter, but the ability to deal with them. The course is not yet a subject for examination. Long may it remain free from the dangers of such treatment. We give no end of the term test in our course. The boys and girls are told at the outset that there will be no test other than the ever-present challenge of every problem that faces them. That this is sufficient to maintain their effort at a high level, and to acquaint the teacher with the relative merits of his pupils, has been borne out by experience.

Many children leave school with a feeling of hopelessness about their mathematics. They are not necessarily stupid, but they have not been able to connect what they have learnt with anything real and tangible, and so they lose confidence in their own powers, and come to look with distrust on this side of their schooling. We hear a good deal about the unmathematical child—too much for our peace of mind, I daresay. Is not the fault largely ours as teachers? I believe that the majority of children are born into the world with the capacity to enjoy number manipulation, just as they are born with an ear for music. If this innate gift is never encouraged, the child comes to be regarded, and to regard himself, as unmusical. Nowadays the spreading of good music by wireless, and the work of enlightened teachers, is helping to make us all better listeners. An equal regard for the mathematical gifts latent in children would prevent the atrophy of powers which normally should enrich the lives of their possessors. We teachers must look to it that this native talent is not stifled.

The boy or girl with a strong mathematical bent will go on from strength to strength, and his needs are probably well catered for by the orthodox course. But the others, for whom the matriculation standard will be forever too formidable an obstacle—are we to condemn them as hopeless? Cannot we find certain fields of mathematical activity, humble in technical difficulty, yet rich in human interest, that may be worthy of their endeavour?—activities that may encourage their belief in their own powers, and may lead them gradually into that habit of mind that faces problems with a certain exactness of thought: the habit of mind that puts hazy ideas to the test of figures; that does not accept opinions without a little checking up by the arithmetic of common-sense; the habit of mind referred to by Francis Galton when he says:

"General impressions are never to be trusted. Unfortunately when they are of long standing they become fixed rules of life, and assume a prescriptive right not to be questioned. Consequently, those who are not accustomed to original enquiry entertain a hatred and horror of statistics. They cannot endure the idea of submitting their sacred impressions to cold-blooded verification. But it is the triumph of scientific men to rise superior to such superstitions, to devise tests by which the value of beliefs may be ascertained, and to feel sufficiently masters of themselves to discard contemptuously whatever may be found untrue."

This scientific temper is what needs to be encouraged. Our problem is to find how this can be done for the unmathematical as well as for the mathematical child. I am sure there are many ways of doing this other than those suggested in this paper. But as a help towards assisting our future citizens to face intelligently the problems that beset them in daily life, I hope schools will give thoughtful consideration to a course on "Arithmetic of Citizenship."

B. L. Gimson.

Discussion.

Miss **M. Punnett** felt the paper should be discussed by someone in a position to criticise and ask questions. Unfortunately she was not in that position in regard to Mr. Gimson's most interesting paper, but the reader would probably be interested to hear that there had for many years been a similar syllabus in use in a demonstration elementary school. Mr. Gimson had, of course, had much better opportunity with older boys and girls of going into what Miss Punnett termed the more advanced parts of the subject, the parts that it really properly led on to. When Sir Percy Nunn devised the syllabus to which she was referring he felt it a pity to have to apply it to an elementary school where it was in use only for boys and girls who left at fourteen, because it was at that age that they began to be keenly interested. It was hoped, with the raising of the school leaving age, to carry on the work to a really proper stage of completion. Not only in elementary but in secondary schools the method outlined was quite the best way of teaching those sorts of subject which in the text-book came under the heading of percentages and which, as Mr. Gimson had shown, were so dull to the children. Boys and girls, even before the age of fourteen, were keenly interested in the arithmetic of citizenship and did not find it too difficult. She hoped Mr. Gimson realised that when a syllabus of that type was put into the First Division Girls' School Report, it was not there because it was supposed to be specially for girls, but because it was a boy and girl matter and was being dealt with as such. It was interesting to note that when the syllabus was first suggested there was objection which would probably amuse some of those present. It was said: "It is silly, and it is not mathematics; we have nothing to do with it." Nowadays people were generally more reasonable and realised that mathematics should be taught as applied to all the sensible obligations of everyday life. In conclusion, the speaker very cordially supported what Mr. Gimson had said with regard to his method being the best way of teaching those particular branches of mathematics which were generally regarded as dull.

Miss **Carless** (Homerton College, Cambridge) endorsed what Miss Punnett and Mr. Gimson had said. She said that a course, almost identical with that outlined, was in operation in the Training College with which she was associated and that she should have assumed that such a course would have been given in most schools, were it not for the fact that the students invariable asked "Why are we not taught this in our schools?" She had been accustomed to teach it in secondary schools, and had also found such lessons a great help in elementary schools. The trouble was, that young teachers had difficulty in getting suitable material from newspapers. However, the students at Homerton College had collectively made an excellent set of

examples, which had been privately published. Such a collection should really consist, not so much of examples, as outline sketches of possible examples, which could be made up to date by the pupils, should arithmetic of citizenship become more regularly taught. Such examples would be most useful in Senior and Central schools.

Miss **Mielziner** (Liverpool) thought most of those present would agree that arithmetic of citizenship was a very necessary subject. What the speaker wanted to know was how present-day teachers were best to prepare themselves for teaching the subject. Probably all who had tried to do so had found it difficult to prepare themselves, though that difficulty would be overcome in about twenty years' time when children at present in schools had been taught the subject. Could Mr. Gimson say how he prepared himself? There did not appear to be a book on the subject under discussion.

Mr. **J. Katz** (Borough School, Croydon) asked if Mr. Gimson could give information with regard to the age of the children he taught, and, secondly, how much of ordinary arithmetic might those children be reasonably supposed to know before they came to him? How did the class link up with other classes in the school? Were the children specially selected C 3 pupils? How many periods did the class have? Information would be helpful in placing the class, as it were, in the geography of the secondary school.

Mr. **Gimson** replied that the age of his pupils ranged from 14 plus to about 17. He had not mentioned age when reading his paper because he did not wish to give the impression that the course was particularly suited to a given age. He felt there was so much good material in it that teachers could adapt it to their classes from 13 plus up to 17, 18 or later. As to the type of children selected, they were of two kinds: either those who, through intelligence tests or otherwise, it was felt had no chance of passing the School Certificate or Matriculation Examinations and therefore had not to take an exterior examination, or those who had already passed the School Certificate or had another couple of years at school before going to the University and thus could spare a little extra time for study of the subject. The former comprised three-quarters of the class. Mr. Gimson regretted that it was not possible, in his own school at any rate, to give a course to all children. He believed it would be worth while if time were available. As it was, it was necessary to meet the requirements of examination and thus the course could only be given to selected pupils. The children had the equivalent of about forty-five minutes per day, excluding Saturdays. No preparation was necessary; it was included in the period. There was a good deal of looking up of data which the children were expected to do as part of their training in the course.

Mr. **Boon** (Dulwich) asked to how many figures of accuracy did Mr. Gimson expect the working of the tables. The working of them involved a good deal of concentration merely for the purpose of accuracy.

Mr. **F. Sandon** (Plymouth) thought there was one correction he might be allowed to make. The cost of living index figures were prepared by the Ministry of Labour and published monthly, whereas Board of Trade figures were wholesale prices.

Mr. **Hope-Jones** (Eton) said he had very much appreciated Mr. Gimson's exceedingly interesting paper and wished he had some opportunity of doing the same job. In connection with the insurance aspect of the subject, did Mr. Gimson get an opportunity of emphasizing the dangers and the fallacies of gambling, so essentially opposed to the subject of insurance?

Mr. **Gimson,** replying to points raised, said that preparation on the part of the teacher responsible for such a course could only be by trial and error. He did not know any text-book which gave the necessary material; the only thing was to begin in a humble way, making mistakes as one went on. So far as he was concerned, every year the material and the method of presentation changed—changed with the classes one had to teach, and, moreover,

one got help from the class as one went along. All learned together—teacher and pupils. Teachers could not possibly be aware of all the facts, and it was helpful for them as well as for the children to feel they were advancing from the unknown to the known. As to the books that might be helpful, he had already mentioned *Whitaker's Almanack*, and in addition there was *The Statesman's Year Book*, containing useful statistical information, together with recent editions of the *Encyclopaedia Britannica*, though the latter soon got out of date. On the whole, books did not give as much help as did the daily newspapers and such pamphlets as one could obtain from insurance companies and so on. As regards accuracy, that was a question of judgment, and of course differed in various classes. One was able to train the judgment of the child to find out what accuracy was suitable in a given case. Of course with money sums one required greater accuracy than with ordinary engineering or science problems. He personally only asked for the kind of accuracy that could be obtained by logarithms or by slide rule. Though few children used the latter, many could, of course, use common logarithms with fair facility. As regards the question of insurance and gambling, that aspect had not occurred to him and, fortunately, not to his class !

A hearty vote of thanks having been accorded to Mr. Gimson for opening the discussion, members adjourned for tea.

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732. Science is simply common sense at its best, that is, rigidly accurate in observation, and merciless to fallacy in logic.—Huxley, *The Crayfish*, 1881, p. 2.

733. Angling may be said to be so like the mathematics, that it can never be fully learnt.—Isaac Walton, *The Complete Angler*, Epistle to the Reader, p. xi., 1875.

734. He who can digest a second or third fluxion, a second or third difference, need not, methinks, be squeamish about any point in divinity.—Berkeley, *The Analyst*, Sect. 7.

735. *Demostración humorística del teorema de Wilson*. To prove by a stroke of the pen that $(p-1)!+1=M \,.\, p$.

This is equivalent to

$$p!-1!+1!=p!=M \,.\, p\,; \;\therefore \text{ etc.}$$

From *Boletin Matematico*, Oct. 1929, p. 150.

736. In the Glasgow University Union a story was told of a professor who referred to an industry which " absorbed one million persons, half of whom were men, and slightly more women." [Per Mr. J. W. Stewart.]

737. Dr. Jameson, of " The Raid " fame, was trying to arrange a tri-weekly mail service between the Cape and Rhodesia. The young Boer with whom he got into touch was already running a waggon every week, but he asserted that though he had not enough rolling stock for a tri-weekly service he would run a bi-weekly service for two-thirds of the money. Jameson readily agreed, and it was not until the contract was signed that he discovered that, while tri-weekly meant three times a week, bi-weekly meant once a fortnight. [Per Mr. J. W. Stewart.]

738. The author thinks he has proved that the moon has no effect upon the tides, and he spends much print in overturning the idea which he supposes the world to entertain, namely, that the acceleration of the moon's motion is proof that she will one day fall upon the earth. . . . The work is but one more of the never-ending instances in which persons who just know enough of mathematics to calculate an eclipse under the direction of others, think they can refute those to whom it is due that astronomers can do something more than calculate an eclipse.—*On Eclipses, etc.*, by T. Kerigan, F.R.S. [*Athenaeum*, 1844, p. 732].

THE MATHEMATICIAN IN ORDINARY INTERCOURSE.

By Miss H. P. Hudson, O.B.E., Sc.D.

I received this morning a pamphlet in which the master of Stowe School, describing the Acropolis, says: "The physical beauty of the place alone would have moved even a mathematician's heart," and that appears to be what the world thinks of us, or, at any rate, a different part of the scholastic world. I am, of course, speaking to mathematicians and not to teachers, and I suppose we have all at one time or another suffered from the misguided hostess who introduces one as a mathematician. When that terrible word is mentioned the other person instantly curls up and says to the so-called mathematician, "How terribly clever you must be," or else, "I never could do maths. at school." Neither is a good opening in conversation, and it takes some time to beat down that barrier and get back to ordinary friendly terms. A mathematical reputation is rather like a clerical collar; it is a hindrance to intercourse and condemns one to a very great deal of loneliness. This may, of course, be partly due to jealousy of those who are academically superior on one particular point, although they are inferior on a great many other points, because there is a certain amount of truth in the gibe that a specialist is a person who has failed in every subject except one, having had neither time nor energy to succeed in the others.

Perhaps it is worth while to look for a few minutes into one or two things in mathematical training that are either handicaps or advantages when it comes to intercourse in other fields. First of all, there is the mathematical language, which I think has rightly been described as the most unintelligible on earth. I remember at college, when three of us were doing research, that the historian could say she was dealing with charters of the reign of King John, which meant something, the bio-chemist that she was seeing how and why geraniums were scarlet, which also meant something, and I could say I was dealing with the fundamental points of Cremona transformations, which meant nothing to anybody. It is a thing one simply cannot share. When people ask you what your last book is about, the best answer you can give, and the one I always give, is: "It took me forty years to find out; if you will give me another twenty, I will explain it." And it is not only in the strict realm of mathematics that our language is unintelligible. There are many quite common objects of general interest for which we mathematicians have beautiful and elegant expressions which, on the score of politeness, we have no right to use. If we want to describe the shape of a hill as a truncated cone we must not say so, because it is just as rude to say that to people who do not know what it means as to talk French or German in the presence of those who do not know either language. It may be true that we do sometimes overstep the bounds of politeness. Besides this great temptation, which is always ours, to use our own language rather than that of other people, we have other habits which are very likely annoying. There are, however, some habits we would perhaps prefer to see more widespread, especially the habit of mental arithmetic and being able to multiply small quantities by large multipliers rapidly and approximately. To most people a penny a day is just a penny a day, but, as a matter of fact, it is something like £100 in a lifetime. A person who comes quickly to a fact such as that is apt to be treated as a magician. I think that is a habit which one may hope eventually to see more widespread. Five minutes is not five minutes but a whole hour wasted if the number on the committee which is kept waiting is twelve, but the average person who turns up late does not think of that beforehand. Mathematicians have no excuse for ever being late.

Take another of our bad habits, which is really a foolish one and is annoying in ordinary intercourse. Certain logical methods that are important and valuable in mathematics are foolish and misleading in topics of general intercourse. For instance, if one point of a mathematical argument is suspected

of fallacy, one pays attention to that point and ignores all the rest till satisfied that that particular link holds good; we are drawn to the weak spot and stress the weakest link. But if the argument is one of politics or economics, and we make any amount of not only weak but false links, so long as some links are good and strong, however weak another link may be, it is not worth while paying attention to it. What is important there is to go for the strongest and not the weakest; to go for the weakest part first is waste of time, and hence people think us the fools that we are.

Another habit which is sensible in mathematics and foolish in other regions is that of dealing with the extreme case. The old Latin tag *extremum probat regulam* means "the extreme case tests the rule," though it is usually mistranslated. One tests a proposition by taking half the coefficients to be zero, and the other half, unity; and if it fails then, one does not consider the general case at all.

If you try to do that in a political argument or in any argument in another region, you are always certain to be applying it to some case for which it was not intended, because, the enunciation not being mathematically expressed, with all the provisos put in, the extremely simple case one takes to test it by is one which would have been definitely ruled out if the enunciation had been accurate. If we use that kind of test, people think we are quibbling. Perhaps we are.

Another bad habit is that of *reductio ad absurdum*, because your hearer thinks it is himself and not a particular proposition which is being made to look absurd, and nobody likes that.

In all these different ways our real mistake is a lack of proportion. It is very difficult, I think, for those who are trained to put first what has logical primacy when going over to other spheres of intercourse to put first what is of importance in other than logical ways; we find it more difficult to come by a true sense of proportion than people with different training.

What first set me off on the somewhat thin subject of this paper was the experience I had of running a series of conferences on co-operative lines with very mixed participants, and it fell to me to have to select certain people for helpers. I found myself picking out mathematicians for certain jobs. It is perfectly true that there are certain services that people with mathematical training can render easily and therefore ought to render, because they do them better and with much less effort than those with different training. There are other jobs that others can do better than mathematicians. No one would ask a mathematician to do the flowers or play for dancing. I think there are limits. There was one society whose annual meeting I gave up attending because for three years running I was asked to scrutinize the ballot and missed the whole of the proceedings!

I think one element in the dislike that mathematicians arouse is that there is supposed to be a certain moral as well as intellectual superiority, which arises from the curious fact, which has no moral value really, that in the sphere of mathematical research one is automatically safeguarded from certain common temptations. There is no sphere in existence in which lying or swank is so unattractive and unprofitable. You are found out directly—you find yourself out—and it does not get you on, in the way it may be supposed to in other spheres. It may be that we poor people do get out of our training, for which we pay a high price, a certain tendency to honesty that comes to us more easily than—well . . . (Laughter).

Another thing from which we are entirely sheltered is avarice, because there is no money in mathematics and therefore that particular root of evil does not come into our universe of discourse. It does not mean we are any stronger morally or any better characters, but simply that for those particular hours that we give to our own work, we have been automatically sheltered from those particular temptations, though of course liable to certain others which

I leave to your imagination. And when we go into the world and come up against those sheltered from other temptations and whose standards are higher than ours in other matters, ought we to expect of them our standards where we are strong ? Ought we not to be prepared to bow to their standards where we are weak ? Do we ask too much of the world when we get annoyed with other people's inferiority, ourselves being beyond reproach ? I think it must be admitted that the world at large ought to take its standards from where they are most easily kept high, and that each profession and each craft has in trust for the universe the safeguarding of certain standards that come easily its way. Perhaps the most important thing that we have in trust for the world is our standard of what you may call truth, if you want to make the most of it, or what you may call mere accuracy, if you want to be scornful. There is an amount of truth in mathematics, from the multiplication table upwards, which does not occur in any other science or art, and I think that is one of the things we have got to guard and, as far as possible, make clear in all its beauty to the rest of the world.

The loneliness of the mathematician or the man with scientific training is generally due, in the long run, to the great gulf fixed between clear-minded and woolly-minded. There are those who see the world black and white and miss all the beautiful colour, and those who see the world through a rainbow fog and doubtless miss all the beauty of form. Like every other gulf in creation, that can be bridged. It takes patience and affection on both sides, but it can be done. And perhaps by and by it may be filled up, when we get education worth calling education, but that will hardly happen in our time. Meanwhile, the great majority of mankind are on the soft side of the gulf, and those on the rocks—well, it is of all the greater importance to have such meetings as these where we can get together and talk our own language amongst ourselves with ease.

Dr. **White** asked why, if there was no money in mathematics, did mathematicians allow non-mathematical people to write mathematical text-books ; there certainly was money in the latter.

Prof. **W. M. Roberts** (Royal Military Academy, Woolwich) said Miss Hudson's remarks had greatly interested him because he believed it good for mathematicians to hear such comments now and again. They should occasionally indulge in flag-wagging, and that reminded him of the toast of a Highland regiment: "Here's to us. Who's like us ? Damned few—and they're all dead." Miss Hudson had said that a mathematician was regarded as a man or woman who lived more or less alone, but, when it came to a job, was that true ? He had always found that a mathematician was expected to know everything. The first post he had taken was that of mathematical lecturer and he had been asked if he could teach Latin. While he had to admit that he had not taken it seriously at school he thought he might learn some. Then he had been asked if he would teach a little Greek, so of course he taught a little Greek. Would a classical man have been asked to teach a little about Fourier's Series ? Certainly not. But why was the mathematical man expected to know everything ? The answer was simple. If one had a certain amount of aptitude for mathematics, one really had the key to all knowledge. A man who was constitutionally unable to put A^2-B^2 into factors had a whole field of knowledge cut clear away from him, for he could read only elementary text-books, and thus half modern progress in knowledge was out of his ken. What was a mathematician cut off from ? It was possible to learn any language if he would take the trouble; he had heard that it was even possible to learn Russian ; the mathematician could read philosophy, if he had patience, and also economics ; he could even understand a book on the gold standard, and perhaps that was why, as he had discovered, the world tacitly assumed that a mathematician was omniscient.

Prof. **H. T. H. Piaggio** suggested that Miss Hudson owed much to excep-

tional circumstances. He believed she was a member of a family nearly every other member of which was a mathematician. An ordinary mathematician had a wife or sister who was a non-mathematician, and thus did not hear much about his virtues.

Mr. **N. F. Sheppard** spoke as one who was not a mathematician, and thus it might be felt that he appeared rather as a wolf in the fold. That, however, was not the case, for he was a Sheppard in wolf's clothing. He had had the opportunity of studying a mathematician at close quarters for nearly a quarter of a century and therefore spoke with some confidence, although he had very little knowledge of the subject itself. Several points had struck him, notably the statement by Miss Hudson that there was a truth in mathematics not found elsewhere. He thought the whole point of mathematics was that the subject had nothing to do with truth, but merely with the connection of one truth with another. All that one could say to one who enquired what one had been doing in mathematical research was that one had been finding out what would be true if . . . (Laughter.) In real life a mathematician suffered from two disadvantages, in that the mathematician had been trained always to act with careful thought and with a logical reason for everything done. The speaker had come to the conclusion that that did not always work in real life. One must not always be logical, because logic was founded upon known facts, and what was called instinct was obedience to things that had been noticed sub-consciously but could not be said to be known. They could not be written down on paper. But if one sat down to act as one thought wisest, without any logical ground, one would find soon that something had been done far cleverer than what had been intended.

He was quite sure that the essence of a wise action was that it should be done not only on logical grounds but also with that curious understanding which was not mere logic.

Again, the mathematician in conversation also was rather different from other people, because in ordinary conversation one must not be too clever. Take, for instance, a group of people dressed for dinner, lightly passing the toy balloon of conversation from one to another and keeping up the spirit of friendly banter, all carefully over-stating and understating their opinions so that there might be something to contradict or room for somebody to add an epigram, as it were, to puncture the balloon. Then came the mathematician. He listened for five seconds ; his eyes were turned inward for fifteen seconds ; then he made a statement and it would be discovered that there was no balloon, and for a long time there would be silence. And so the mathematician, for those reasons, was not greatly liked.

Mr. **Wright** (Winchester) thought one of the few things mathematicians had to learn was not to generalize from single instances, and wondered whether the last speaker had learned that lesson. He very much doubted whether, as a whole, when the subject was under discussion and they were talking about *the* mathematician, they were not all making the same mistake and making generalizations which, amusing as they were, might be somewhat difficult to prove.

Mr. **N. F. Sheppard**, in justice to the President, explained that he had not spoken from one experience.

Prof. **Alfred Lodge** said he understood that Lord Cromer always chose mathematicians for administrative work in Egypt on the ground that their business was to solve problems, certainly problems of one kind, and he had no hesitation in setting them administrative problems, because he knew they would tackle them in the same spirit as they tackled mathematical problems. He believed he was right in saying that Lord Cromer was never " let down."

Mr. **J. Katz** (Croydon) said he had recently read a book with the intriguing title *The Psychology of Philosophers.* He thought it might be claimed that Miss Hudson had made a contribution to that very fascinating subject,

"The Psychology of Original Mathematicians." Looking through the history of mathematics, it was rather interesting, if not surprising, to find how many mathematicians were of a very grave and somewhat melancholy disposition. There came to his mind Archimedes, Lagrange and Newton, to take three outstanding examples. Perhaps something similar happened in the case of philosophy; at any rate the writer of the book he had cited thought philosophers tended to be disappointing in love and disappointed in love; that they did not appear to have the knack of getting on with their fellow men, and that they soon found it necessary to retire into themselves and there build up a private realm of their own in which their faculties could obtain the satisfaction of free activity. And so it might be that the original mathematician was in the same class as the philosopher, and that he, too, found it necessary to retire into a private realm of his own where he could construct the beautiful figures of mathematics. He wondered whether the primary satisfaction of the original mathematician was not really that of an artistic kind: the satisfaction ultimately of something of an aesthetic character, and the fact that his realm was cut off from the impure world of concrete application tended to add to that satisfaction, for there nobody could get at him. That was a realm in which he was perfectly free, and where, if he succeeded, the mathematician could obtain all the recognition he needed. The speaker wondered also whether Plato was not really responsible for some of the misfortunes of pure mathematicians, for Plato rather prided himself on the tradition that the philosopher, the thinker, or the mathematician was a person who lived a cloistered life away from the mere vulgar activities of men. He wondered, for example, whether Plato would have approved of the lecture that Professor Roberts had given during the morning session, in which he dealt with vulgar things such as guns, and did not deal with certainty but with mere probability. He fancied Plato would have been much more likely to approve of the lecture by Professor Chapman, if only because the latter spoke of beautiful things which satisfied the high Platonic faculties, namely, spherical harmonics. The Platonic tradition had come down through the ages owing to the fact that so much of learning had been attributable to monkish schools, and the tradition had been carried on in the present day by clerical headmasters, so that one found all kinds of forces and powers at work to influence the character of a pure mathematician's existence. In that connection there occurred to the speaker a passage in a recent number of the *Church Times* in which it was suggested that impure or applied mathematicians tended to be rather free-thinkers and infidels, whereas all pure mathematicians were Anglo-Catholics! Those who were just common or garden teachers of mathematics did not suffer the same kind of torture as the pure mathematicians; the former were up against their pupils, who would bring them down to earth in quick time. Undoubtedly the fact that they were in contact with young people put a different complexion on the sort of life teachers of mathematics led. Another point with regard to the latter: if they really were going to be good teachers they had to have some kind of philosophy about the significance of their own subject. It was not possible, year in year out, to go on drumming the same sort of stuff into generation after generation of schoolboys or schoolgirls unless the teacher was convinced that he or she was really imparting something of fundamental value. He thought there were moments of dubiety, moments of considerable scepticism, when one wondered was the subject really worth while; was it worth while trying to push that sort of subject into those particular children? Thus he thought it a very good thing for teachers of mathematics occasionally to contemplate the philosophy of their subject in order that they might have the strength to go on with the daily work.

Mr. **Hope-Jones** (Eton) reminded the audience that Miss Hudson had questioned whether mathematicians were the stronger morally for the

standard of truth they set themselves. He could give a very definite positive answer: the mathematician did his work wrongly (if he happened to be the speaker); if he was not satisfied then he went on until he put it right. The mathematician had the habit that perhaps the scientist and others had of going back on preconceived notions when it was found they would not work. The speaker believed that all religious, political and sociological life and thought would be infinitely improved, to the benefit of the world, if all the community could learn the mathematician's standard of criticism and willingness to go back on preconceived notions. It was hardly to be expected that he would not butt in with "probability," and so he contended that if one excluded "probability" from mathematics it became too hard and fast to apply to the problems of real life. As Miss Hudson said, some saw things in black and white and they were not so. But as a result of the inclusion of "probability" and the allied study of correlation, which he had compared to "laws out of focus," the mathematician was closely in touch with the world as it was governed. Human life was not governed like mechanical life by hard and fast laws which could be clearly put down in equations, but by general tendencies, which could be called "laws out of focus." Lives were moulded on probability. He believed that the mathematician had a better chance than others of seeing something of those principles and consequently moulding his life and thought more in accordance with actual facts.

Dr. **W. Stott** thought it might be taken that a meeting of the Mathematical Association could be expressed in mathematical terms as a compensation to singularities, but he did not admit that the study of mathematics had hindered or in any way affected the pleasure that a mathematician might take in society or conversation. It had been his privilege to meet a good many students of advanced mathematics, and in all cases he had found that they had been better conversationalists than lawyers and even actors. He knew one mathematician, and a very good one, a German, who was a musician and a linguist. He had had four Senior Wranglers in his house at one time and never had he been with a more interesting group of conversationalists. Then looking at mathematicians as a whole, there came to his mind names such as Dr. Clifford, a great conversationalist; Professor Hobson and many other men who were well known as mathematicians but who knew other subjects well. Moreover, he had been for many years secretary of the Liverpool Mathematical Society, and it had been his business to get lecturers and visitors to attend meetings, and he noticed that excuses given for inability to be present were quite those of ordinary people: there was bridge, or golf, or something of the sort. Finally, he thought it would be found, when one looked into the question, that the average conversation of the mathematician was better than the average conversation of almost any other learned profession.

Dr. **J. T. Combridge** thought Mr. Sheppard would agree that he had seen a toy balloon inflated and lost by mathematicians after having been successfully tossed about for three-quarters of an hour. He would like to remind Mr. Sheppard of Mr. Bertrand Russell's remark that "mathematics is the science of a subject in which we do not know what we are talking about, and do not care whether what we say is true." That probably applied to the spirit in which most had entered into the discussion. He would also like to say that if Mr. Sheppard met with a better description of an ordinary dinner-time conversation in non-mathematical circles he personally would like to be acquainted with it.

The **President**: I will now ask Dr. Hudson to reply.

Dr. **Hudson**: I think the bubble is burst!

The **President**: Then we have only now to thank Dr. Hudson very much for having introduced this subject and for having supplied a very pleasant close to these two days' meetings.

EUCLID (I. 4) AND TIME-SPACE THEORY.

A REPLY TO Mr. E. T. DIXON, M.A.

By Alfred A. Robb, Sc.D., F.R.S.

I have read with interest Mr. E. T. Dixon's criticism of my paper on a " Partial Failure of Euclid (I. 4) in Time-Space Theory." Mr. Dixon takes objection to this title, although, in his own words, he admits that: " Time-Space Theory may, in a limiting instance, be incompatible with Euclid's (I. 4)."

I think I may safely leave it to readers of *The Mathematical Gazette* to decide which form of words they prefer, but, if I were in a mood for quibbling, I think I might take objection to Mr. Dixon's words on a very similar ground to one on which he takes objection to mine; namely, that, as Euclid was not dealing with Time-Space Theory, his proposition (I. 4) could not be incompatible with it.

Mr. Dixon appears to think that I am sneering at Euclid in speaking of a *partial failure* of his proposition (I. 4). Mr. Dixon surely ought to know that to speak of the " failure " of a mathematical proposition in a particular case is a common mode of expression; meaning that, in such case, the proposition does not hold, and that to speak of a partial failure of Euclid (I. 4) refers to the proposition and not to the man.

Very far from desiring to sneer at Euclid, I have the greatest admiration for his work, and only wish that some of our modern mathematicians were as careful; but I do not consider it any disparagement to suggest, after more than two thousand years, that certain difficulties which he encountered might be cleared up and that certain emendations are desirable.

As regards the *method of superposition* I should like to call Mr. Dixon's attention to Sir T. L. Heath's comments on this subject in his monumental edition of *Euclid's Elements* (see vol. i. pages 225 and 249). On page 249 he says: " It may be that Euclid himself was as well aware of the objections to the method as are his modern critics; but at all events those objections were stated, with almost equal clearness, as early as the middle of the sixteenth century. Peletarius (Jacques Peletier) has a long note on this proposition (*In Euclidis Elementa geometrica demonstrationum libri sex*, 1557), in which he observes that, if superposition of lines and figures could be assumed as a method of proof, the whole of geometry would be full of such proofs, that it could equally well have been used in I. 2, 3 (thus in I. 2 we could simply have supposed the line taken up and *placed* at the point), and that in short it is obvious how far removed the method is from the dignity of geometry."

Sir T. L. Heath also remarks (p. 225) that: " The phraseology of the propositions, *e.g.* I. 4 and I. 8, in which Euclid employs the method indicated, leaves no room for doubt that he regarded one figure as actually *moved* and *placed upon* the other."

In case Mr. Dixon still regards superposition as a valid method of proof, I would suggest that instead of drawing his triangles on paper, he draw them on thin sheets of india-rubber and then proceed with his superpositions and consider their cogency as proofs of geometrical theorems.

Doubtless Mr. Dixon will say that india-rubber is extensible. Quite so; but find me some solid material which is not!

India-rubber only possesses in an exaggerated degree a property of extensibility common to all solid bodies.

The most rigid bodies we possess are apt to stretch to an extent which, though negligible in most of the affairs of every-day life, may be of the utmost importance in some of the very accurate observations of modern science (*e.g.* the Michelson-Morley experiment), and it behoves one to attach definite meanings to constancy and equality of magnitudes.

Mr. Dixon says: "Unfortunately pure mathematicians as such do not trouble themselves much about the real import of their symbols."

I am afraid that Mr. Dixon has been falling into the evil ways of the "pure mathematicians as such," inasmuch as he does not ascribe any definite meaning to that *constancy of magnitude* which the method of superposition presupposes.

That *a given solid body is of the same dimensions when in position B as it was when in position A* is a statement which may be true or may be false, but is certainly far removed from being a mere tautology; as it would be if there were no other meaning attached to *congruence* than that derived from the properties of that body itself.

There can be no doubt, however, that the properties of the least extensible bodies do provide a *crude* conception of *congruence* which serves for most of the purposes of our daily life, but is quite inadequate for the more accurate work of modern science.

Strange as it may appear to Mr. Dixon, my *Theory of Time and Space* was written with the object of developing a logical theory of *congruence* which, in its practical application to the physical world, should depend upon something more fundamental than the properties of approximately inextensible bodies.

This more fundamental basis was found in the *before* and *after* relations, which served both as a ground for a pure mathematical structure, and, at the same time, had a definite interpretation in our experience. It is, therefore, quite incorrect to say, as Mr. Dixon does, that my avoidance of any mention of rigid bodies implies that my theory is a purely symbolic one.

The postulates * of my theory were all expressed either directly in terms of these *before* and *after* relations, or else indirectly so: that is to say, through the medium of words already defined in terms of these relations.

Thus, provided that there be physical facts corresponding to these postulates (and there appear to be optical ones which do so, at least, to a very close approximation), there must be a physical basis for the whole structure.

The logical superstructure (which is, of course, independent of the physical basis) is fairly complicated, but I believe it is sound, and the theory has the great advantage that it does not involve either measuring rods or clocks and enables us to prove theorems which the early mathematicians dealt with only by superposition.†

The geometry which I develop in my book is four-dimensional in character; but one of these dimensions is of a different nature from the other three and, *of this system, ordinary Euclidean geometry forms a part.*

The usual treatment of the foundations of Euclidean geometry is based on the three-term relation of *between*; whereas this system is based on a two-term asymmetrical relation of *after*.

In the usual treatment *congruence* appears as if it were something extraneous grafted on to an otherwise complete system; whereas in my theory, *congruence* forms an intrinsic part of the whole. This difference is due to its being built up from a two-term relation instead of a three-term one.

In my opinion, if Euclid could have seen how to build up his geometrical system without making use of the method of superposition he would have done so; for, as Sir T. L. Heath remarks (p. 225): "it is clear that Euclid disliked the method and avoided it whenever he could, *e.g.* in I. 26"; while Mr. Bertrand Russell says that Euclid "would have done better to assume this proposition (I. 4) as an axiom."

But to proceed with an examination of Mr. Dixon's criticism. On page 3 he remarks: "And I am bound to say that, though Dr. Robb's symbolic

* In view of what he says, Mr. Dixon should observe that I use the word *postulates* not *axioms*.

† Thus in Theorem 189 of my book will be found the proof of the analogue of (I. 4) for the case of right-angled triangles which does not involve any superposition.

analysis is quite all right, when he comes to give it real import, he here makes what Aristotle might have called a fallacy of the ambiguous middle. Referring to his diagram, he says that 'the triangle' (OQP) 'is not determined in all its dimensions when the magnitudes of the sides OQ and QP are given, along with the fact that they are normal to one another.' This is supposed to indicate a 'failure' of Euclid's I. 4; it must, therefore, be taken that Dr. Robb mentally substituted for his term 'normal' an Euclidean term, such as 'at right angles,' with the Euclidean implication that all right-angles are equal to one another. But the way Dr. Robb appears to define his 'normal' is not at all the same as Euclid's definition of a right-angle; which, in effect, is by superposition."

With regard to this passage I may say that I used the word "*normal*" here deliberately, and, since Mr. Dixon says that he has access to my *Theory of Time and Space*, I would venture to call his attention to a passage on page 204 where, in defining the "normality" of the different types of line I say: "Only one case will be found to be strictly analogous to the normality of intersecting straight lines in ordinary geometry; namely the case of two separation lines. The other cases are so different from our ordinary ideas of lines '*at right angles*' that we do not propose to use this expression in connection with them. Thus any optical line is regarded as being 'normal to itself' and the use of the words 'at right angles' would, in this case, clearly be an abuse of language." In the face of this passage does Mr. Dixon still imagine that I "mentally substituted for (my) term 'normal' an Euclidean term, such as 'at right-angles'"?

I have had far too much experience of the complication which arises from the "normality" of the different types of lines to make such a mistake.

In this particular instance I was dealing with the "normality" of an inertia line to a separation line, and I adhered strictly to the principle enunciated in the passage quoted from my book, and did not say that the two were at right-angles. Now although, as I stated in that passage, such a case of normality is not *strictly* analogous to the normality of intersecting straight lines in ordinary geometry, yet there is *a certain degree* of analogy, as otherwise it would not be worth while to use the word *normal* in both cases. In this case the words I actually employed were: "so that the *analogue* of Euclid (I. 4) does not hold in this case."

I did not, however, confine myself to the consideration of the case where the two given sides are *normal* to one another; but went on from that to treat the case where the two given sides are both segments of inertia lines of lengths b and c and making a finite hyperbolic angle with one another equal to $\log \frac{b}{c}$.

I showed that, in this case too, the analogue of Euclid (I. 4) does not hold; so that instead of merely being able to say that the analogue breaks down in a certain case when the given pairs of sides are "normal," I could say that it also breaks down in certain cases when the given pairs of sides are at the appropriate hyperbolic angles.

Again Mr. Dixon says: "Well then, in Dr. Robb's Time-Space diagram you *could not* rotate QP round Q to QR."

I never claimed to be able to do so; and I will go further and say that Mr. Dixon *could not* rotate any geometrical line from one position to another. At best he could rotate a material body, which is an entirely different thing. To do so would be tantamount to changing the position of a position.

A. A. Robb.

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739. My friends are not great arithmeticians, but they are good bookkeepers.—Sir Walter Scott.

THE POWER SERIES AND THE INFINITE PRODUCTS FOR $\sin x$ AND $\cos x$.

BY PROF. H. S. CARSLAW, Sc.D.

1. The course of Plane Trigonometry "up to and including the solution of triangles and the properties of the circles associated with a triangle" offers little difficulty to the teacher and no serious difficulty to the pupil. But in the further development of the subject, when the circular functions $\sin x$, $\cos x$, ... are regarded as functions of the real variable x, the position is different.

Text-books meant to take the pupil up to the standard of the Higher School Certificate or University Entrance Scholarship Examinations usually include the fundamental theorems :

$$\left.\begin{aligned}\sin x &= x - \frac{x^3}{3!} + \ldots = \sum_0^\infty (-1)^r \frac{x^{2r+1}}{(2r+1)!}\\ \cos x &= 1 - \frac{x^2}{2!} + \ldots = \sum_0^\infty (-1)^r \frac{x^{2r}}{(2r)!}\end{aligned}\right\}. \quad \text{The Power Series.}$$

$$\left.\begin{aligned}\sin x &= x \prod_1^\infty \left(1 - \frac{x^2}{r^2\pi^2}\right)\\ \cos x &= \prod_1^\infty \left(1 - \frac{2^2x^2}{(2r-1)^2\pi^2}\right)\end{aligned}\right\}. \quad \text{Infinite Products.}$$

In some of them it is pointed out that the demonstrations given are incomplete. In others the demonstrations are supposed to be rigorous, but are far from satisfactory. In only a few advanced texts can the proof be said to be sound.

Often the discussion is complicated by the introduction of the complex variable, a procedure which, it seems to me, should at this stage be avoided.

It does not seem to be generally known that a rigorous proof of these four properties offers no serious difficulty, even when the real variable only is used. And this paper, which lays no claim to originality but embodies the plan followed by me with my classes for several years, for that reason may be useful. It is obvious that we need only prove them for positive values of x.

2. *The Power Series for sin x and cos x.*

Of course the Power Series for $\sin x$ and $\cos x$ follow at once from Maclaurin's Series, and a rigorous proof of Maclaurin's Theorem, applicable to these functions, does not require more than the elements of the Differential Calculus usually taken by pupils at this stage. But a "proof" which assumes that $\sin x$ or $\cos x$ can be expanded in a power series and that the series can be differentiated term by term is no proof at all.

However, using the Differential Calculus, there is another method which should be more widely known.* And a few lines will be given to it before passing on to the general method applicable to all four of the properties named in § 1.

Let $\quad S_1 = \sin x - x, \qquad C_1 = \cos x - 1,$

$$S_2 = \sin x - x + \frac{x^3}{3!}, \qquad C_2 = \cos x - 1 + \frac{x^2}{2!},$$

and so on.

Thus, for any positive integer n, we have

$$S_n = \sin x - \sum_0^{n-1} (-1)^r \frac{x^{2r+1}}{(2r+1)!}, \quad C_n = \cos x - \sum_0^{n-1} (-1)^r \frac{x^{2r}}{(2r)!}.$$

Also S_n and C_n both vanish when $x = 0$.

* Cf. Bromwich, *Infinite Series* (2nd ed.), § 59.

But $\frac{d}{dx} S_1 = C_1$, which is zero when x is zero or a multiple of 2π and negative for all other positive values of x.

Thus $$S_1 < 0, \text{ when } x > 0.$$

But $$\frac{d}{dx} C_2 = -S_1 > 0, \text{ when } x > 0.$$

Thus $$C_2 > 0, \text{ when } x > 0.$$

Again $$\frac{d}{dx} S_2 = C_2 > 0, \text{ when } x > 0.$$

Thus $$S_2 > 0, \text{ when } x > 0.$$

In this way we see that, when $x > 0$, S_n and C_n are positive when n is even, and negative when n is odd.

Also the following inequalities hold, when $x > 0$:

$$\left.\begin{array}{l} \sum_0^{2n} (-1)^r \frac{x^{2r+1}}{(2r+1)!} > \sin x > \sum_0^{2n+1} (-1)^r \frac{x^{2r+1}}{(2r+1)!} \\ \sum_0^{2n} (-1)^r \frac{x^{2r}}{(2r)!} > \cos x > \sum_0^{2n+1} (-1)^r \frac{x^{2r}}{(2r)!} \end{array}\right\} . \qquad \text{............(1)}$$

But the series $\sum_0^{\infty} (-1)^r \frac{x^{2r+1}}{(2r+1)!}$ and $\sum_0^{\infty} (-1)^r \frac{x^{2r}}{(2r)!}$, are convergent for all values of x.

Therefore, letting $n \to \infty$ in (1), we have *

$$\left.\begin{array}{l} \sum_0^{\infty} (-1)^r \frac{x^{2r+1}}{(2r+1)!} \geqq \sin x \geqq \sum_0^{\infty} (-1)^r \frac{x^{2r+1}}{(2r+1)!} \\ \sum_0^{\infty} (-1)^r \frac{x^{2r}}{(2r)!} \geqq \cos x \geqq \sum_0^{\infty} (-1)^r \frac{x^{2r}}{(2r)!} \end{array}\right\} . \qquad \text{..............(2)}$$

It follows that the sign of equality must be taken, and the Power Series for $\sin x$ and $\cos x$ are established.

3. The proof just given is probably the shortest and easiest way of establishing these Power Series. But if we first prove Tannery's Theorem,† all four relations can be obtained without difficulty. The significance of this theorem Bromwich seems to have been the first to appreciate: and again I refer to his *Infinite Series*, a book which it would be almost impossible to praise too highly.

TANNERY'S THEOREM. *Let $F(n)$ be the sum of n terms each depending on n : e.g.*

$$F(n) = v_1(n) + v_2(n) + \ldots + v_n(n). \qquad \text{.....................(1)}$$

Also let $$\lim_{n\to\infty} v_r(n) = w_r, \ r \text{ being fixed.} \qquad \text{.................................(2)}$$

And let $$|v_r(n)| \leqq M_r, \text{ when } r = 1, 2, \ldots n, \qquad \text{.....................(3)}$$

where $$\sum_1^{\infty} M_r \text{ is a convergent series of positive constants.} \qquad \text{..............(4)}$$

Then $$\sum_1^{\infty} w_r \text{ converges and } \lim_{n\to\infty} F(n) = \sum_1^{\infty} w_r.$$

We are given that $\lim_{n\to\infty} v_r(n) = w_r$, when r is any fixed positive integer.

If follows that $$\lim_{n\to\infty} |v_r(n)| = |w_r|.$$

* If $u_n > a$ and $\lim u_n$ exists, then $\lim u_n \geqq a$.

† This theorem was given by J. Tannery in his *Introduction à la Théorie des Fonctions d'une Variable* (2e éd., Paris, 1904), p. 292. See Bromwich, *loc. cit.*, § 49.

But $$|v_r(n)| \leqq M_r \text{ by (3).}$$

Therefore $$\lim_{n\to\infty} |v_r(n)| \leqq M_r.$$

Thus $$|w_r| \leqq M_r \quad \text{and} \quad \sum_1^\infty w_r \text{ converges.} \qquad \text{(5)}$$

Again by (4), to the arbitrary positive number ϵ there corresponds a positive integer ν, such that

$$M_{n+1} + M_{n+2} + \dots \text{ to } \infty < \epsilon, \text{ when } n \geqq \nu. \qquad \text{(6)}$$

And by (2), there are positive integers $n_1, n_2, \dots n_\nu$, such that

$$\left.\begin{aligned} |v_1(n) - w_1| &< \frac{\epsilon}{\nu}, \text{ when } n \geqq n_1,\\ |v_2(n) - w_2| &< \frac{\epsilon}{\nu}, \text{ when } n \geqq n_2,\\ |v_\nu(n) - w_\nu| &< \frac{\epsilon}{\nu}, \text{ when } n \geqq n_\nu.\end{aligned}\right\} \qquad \text{(7)}$$

Let N be the largest of the integers $\nu, n_1, n_2, \dots n_\nu$.

Now $$F(n) - \sum_1^\infty w_r = \left[\sum_1^n v_r(n) - \sum_1^\nu v_r(n)\right] + \sum_1^\nu [v_r(n) - w_r] - \sum_{\nu+1}^\infty w_r.$$

Therefore $$\left|F(n) - \sum_1^\infty w_r\right| \leqq \sum_{\nu+1}^n |v_r(n)| + \sum_1^\nu |v_r(n) - w_r| + \sum_{\nu+1}^\infty |w_r|. \qquad \text{(8)}$$

But by (3) and (6), $$\sum_{\nu+1}^n |v_r(n)| \leqq \sum_{\nu+1}^n M_r < \epsilon, \text{ when } n \geqq N.$$

And by (7), $$\sum_1^\nu |v_r(n) - w_r| < \epsilon, \text{ when } n \geqq N.$$

Also by (5), $$\sum_{\nu+1}^\infty |w_r| < \epsilon.$$

It follows from (8), that $\left|F(n) - \sum_1^\infty w_r\right| < 3\epsilon$, when $n \geqq N$.

Thus $$\lim_{n\to\infty} F(n) = \sum_1^\infty w_r.$$

Cor. I. It is clear that this argument will still hold, if we replace (3) in the statement of the theorem by the condition that $|v_r(n)| \leqq M_r$, when $r \geqq r_0$, where r_0 is a fixed positive integer.

Cor. II. The theorem also holds when $F(n) = v_1(n) + v_2(n) + \dots + v_p(n)$, and p is a positive integer depending on n and tending to ∞ with n.

4. We know from De Moivre's Theorem that, if n is any positive integer,

$$\sin(2n+1)\theta = \cos^{2n+1}\theta\left[(2n+1)\tan\theta - \frac{(2n+1)2n(2n-1)}{3!}\tan^3\theta + \dots \text{ to } (n+1) \text{ terms}\right]$$

and $$\cos(2n+1)\theta = \cos^{2n+1}\theta\left[1 - \frac{(2n+1)2n}{2!}\tan^2\theta + \dots \text{ to } (n+1) \text{ terms}\right].$$

Let x be any positive number and write $(2n+1)\theta = x$.

Then

$$\sin x = \cos^{2n+1}\frac{x}{2n+1}\sum_0^n (-1)^r \frac{(2n+1)(2n)\dots(2(n-r)+1)}{(2r+1)!}\tan^{2r+1}\frac{x}{2n+1} \qquad (1)$$

and $$\cos x = \cos^{2n+1}\frac{x}{2n+1}\sum_0^n (-1)^r \frac{(2n+1)(2n)\dots(2(n-r)+2)}{(2r)!}\tan^{2r}\frac{x}{2n+1}. \qquad (2)$$

Let $v_r(n) = (-1)^r \dfrac{(2n+1)(2n)\dots(2(n-r)+1)}{(2r+1)!} \tan^{2r+1} \dfrac{x}{2n+1}$. ....(3)

Then $v_r(n) = (-1)^r \dfrac{\left(1-\dfrac{1}{2n+1}\right)\left(1-\dfrac{2}{2n+1}\right)\dots\left(1-\dfrac{2r}{2n+1}\right)}{(2r+1)!}$

$$\left[(2n+1)\tan\frac{x}{2n+1}\right]^{2r+1}. \quad\text{.......(4)}$$

Put $$F(n) = \sum_0^n v_r(n).$$

Then $$\sin x = \cos^{2n+1}\frac{x}{2n+1} \,.\, F(n). \quad\text{..................(5)}$$

But from (4) it is clear that, when r is fixed,

$$\lim_{n\to\infty} v_r(n) = (-1)^r \frac{x^{2r+1}}{(2r+1)!}. \quad\text{..........................(6)}$$

Also, if x is a given positive number, we can choose a positive integer m such that $(2m+1)\dfrac{\pi}{2} > x$.

Thus, when $n > m$, we have

$$0 < \frac{x}{2n+1} < \frac{x}{2m+1} < \tfrac{1}{2}\pi.$$

But $\dfrac{\tan\phi}{\phi}$ continually increases as ϕ passes from 0 to $\frac{1}{2}\pi$.*

Therefore $$0 < (2n+1)\tan\frac{x}{2n+1} < (2m+1)\tan\frac{x}{2m+1}. \quad\text{..................(7)}$$

And, when $n > m$, from (4) and (7), we have

$$|v_r(n)| < \frac{\xi^{2r+1}}{(2r+1)!}, \text{ where } \xi = (2m+1)\tan\frac{x}{2m+1}. \quad\text{..............(8)}$$

But the series $\displaystyle\sum_0^\infty \frac{\xi^{2r+1}}{(2r+1)!}$ is a convergent series of positive constants.

Thus all the conditions of Tannery's Theorem are satisfied.

Therefore $$\lim_{n\to\infty} F(n) = \sum_0^\infty (-1)^r \frac{x^{2r+1}}{(2r+1)!}. \quad\text{............................(9)}$$

But it is known † that $\displaystyle\lim_{n\to\infty} \cos^{2n+1}\frac{x}{2n+1} = 1.$ ....(10)

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* This can be proved easily by the Differential Calculus; and a proof without the Calculus is to be found in Hobson's *Plane Trigonometry* (7th ed.), p. 128.

† We know that $$0 < \frac{-\log(1-x)}{x} = 1 + \frac{x}{2} + \frac{x^2}{3} + \dots, \text{ when } 0 < x < 1,$$
$$< 1 + \frac{1}{2^2} + \frac{1}{2^3} + \dots, \text{ when } 0 < x < \tfrac{1}{2}.$$
$$< \tfrac{3}{2}.$$

Therefore $$|\log(1-x)| < \tfrac{3}{2}x, \text{ when } 0 < x < \tfrac{1}{2}.$$

Now let $$y = \cos^n\frac{x}{n}.$$

Then $$\log y = \frac{n}{2}\log\left(1 - \sin^2\frac{x}{n}\right).$$

Therefore $$|\log y| < \frac{3n}{4}\sin^2\frac{x}{n}, \text{ when } \sin^2\frac{x}{n} < \frac{1}{2}$$
$$< \frac{3x^2}{4n}.$$

Thus $$\lim_{n\to\infty} \log y = 0 \quad\text{and}\quad \lim_{n\to\infty} y = 1.$$

Thus from (5), we have

$$\sin x = \sum_0^\infty (-1)^r \frac{x^{2r+1}}{(2r+1)!}, \text{ when } x > 0.$$

The corresponding theorem for $\cos x$ follows in the same way from (2).

5. We know that

$$x^{2n+1} - a^{2n+1} \equiv (x-a) \prod_1^n \left(x^2 - 2ax \cos \frac{2r\pi}{2n+1} + a^2\right). \quad \dots\dots\dots(1)$$

Therefore

$$\left(\frac{x}{a}\right)^{n+\frac{1}{2}} - \left(\frac{a}{x}\right)^{n+\frac{1}{2}} = 2^n \left[\left(\frac{x}{a}\right)^{\frac{1}{2}} - \left(\frac{a}{x}\right)^{\frac{1}{2}}\right] \prod_1^n \left(\frac{1}{2}\left[\left(\frac{x}{a}\right) + \left(\frac{a}{x}\right)\right] - \cos \frac{2r\pi}{2n+1}\right). \dots(2)$$

Now put $x = a(\cos 2\theta + i \sin 2\theta)$, and we have

$$\sin (2n+1)\theta = 2^n \sin \theta \prod_1^n \left(\cos 2\theta - \cos \frac{2r\pi}{2n+1}\right).$$

Therefore $$\frac{\sin (2n+1)\theta}{\sin \theta} = 2^{2n} \prod_1^n \left(\sin^2 \frac{r\pi}{2n+1} - \sin^2\theta\right). \quad \dots\dots\dots(3)$$

Let $\theta \to 0$, and we see that

$$(2n+1) = 2^{2n} \prod_1^n \sin^2 \frac{r\pi}{2n+1}. \quad \dots\dots\dots(4)$$

Thus from (3) and (4), we have

$$\frac{\sin (2n+1)\theta}{(2n+1)\sin \theta} = \prod_1^n \left(1 - \frac{\sin^2\theta}{\sin^2 \frac{r\pi}{2n+1}}\right). \quad \dots\dots\dots(5)$$

In the same way, from the identity

$$x^{2n+1} + a^{2n+1} \equiv (x+a) \prod_1^n \left(x^2 - 2ax \cos \frac{2r-1}{2n+1}\pi + a^2\right), \quad \dots\dots\dots(6)$$

we have $$\frac{\cos (2n+1)\theta}{\cos \theta} = \prod_1^n \left(1 - \frac{\sin^2\theta}{\sin^2 \frac{2r-1}{2n+1}\frac{\pi}{2}}\right). \quad \dots\dots\dots(7)$$

6. *The Infinite Products for $\sin x$ and $\cos x$.*

Obviously no proof is needed, when x is zero or a multiple of π in the case of the sine, or an odd multiple of $\frac{1}{2}\pi$ in the case of the cosine. And if the results hold for positive values of x, they are also true for negative values.

From (5) of § 5, we have

$$\frac{\sin x}{(2n+1)\sin \frac{x}{2n+1}} = \prod_1^n \left(1 - \frac{\sin^2 \frac{x}{2n+1}}{\sin^2 \frac{r\pi}{2n+1}}\right), \quad \dots\dots\dots(1)$$

and from (7) of § 5, $$\frac{\cos x}{\cos \frac{x}{2n+1}} = \prod_1^n \left(1 - \frac{\sin^2 \frac{x}{2n+1}}{\sin^2 \frac{2r-1}{2n+1}\frac{\pi}{2}}\right). \quad \dots\dots\dots(2)$$

Now let x be any positive number not a multiple of π, and take n so large that $x/(2n+1)$ is less than π.

Also let * $$v_r(n)=\log\left|1-\frac{\sin^2\dfrac{x}{2n+1}}{\sin^2\dfrac{r\pi}{2n+1}}\right|. \qquad\text{(3)}$$

Then $$\log|\sin x|-\log\left[(2n+1)\sin\frac{x}{2n+1}\right]=F(n), \qquad\text{(4)}$$

where $$F(n)=\sum_1^n v_r(n).$$

It is clear that $\lim\limits_{n\to\infty} v_r(n)=\log\left|1-\dfrac{x^2}{r^2\pi^2}\right|$, r being fixed. (5)

Also $\dfrac{\phi}{\sin\phi}$ increases from 1 to $\frac{1}{2}\pi$, as ϕ passes from 0 to $\frac{1}{2}\pi$.†

Thus $$0<\frac{\dfrac{r\pi}{2n+1}}{\sin\dfrac{r\pi}{2n+1}}<\tfrac{1}{2}\pi, \text{ when } r=1, 2, \ldots n.$$

Therefore $$0<\frac{1}{(2n+1)^2\sin^2\dfrac{r\pi}{2n+1}}<\frac{1}{4r^2}, \text{ when } r=1, 2, \ldots n.$$

But $$0<(2n+1)^2\sin^2\frac{x}{2n+1}<x^2.$$

Therefore $$0<\frac{\sin^2\dfrac{x}{2n+1}}{\sin^2\dfrac{r\pi}{2n+1}}<\frac{x^2}{4r^2}, \text{ when } r=1, 2, \ldots n. \qquad\text{(6)}$$

It follows that there is a positive integer m depending on x, such that

$$\frac{\sin^2\dfrac{x}{2n+1}}{\sin^2\dfrac{r\pi}{2n+1}}<\tfrac{1}{2}, \text{ when } m\leqq r\leqq n.$$

Therefore $$|v_r(n)|=\left|\log\left(1-\frac{\sin^2\dfrac{x}{2n+1}}{\sin^2\dfrac{r\pi}{2n+1}}\right)\right|<\frac{3x^2}{8r^2}, \text{ when } m\leqq r\leqq n. \qquad\text{(7)}$$

Thus all the conditions of Tannery's Theorem are satisfied, and

$$\lim_{n\to\infty} F(n)=\sum_1^\infty \log\left|1-\frac{x^2}{r^2\pi^2}\right|$$
$$=\lim_{n\to\infty}\log\prod_1^n\left|1-\frac{x^2}{r^2\pi^2}\right|. \qquad\text{(8)}$$

Therefore from (4) and (8), we have

$$\log\frac{|\sin x|}{x}=\lim_{n\to\infty}\log\prod_1^n\left|1-\frac{x^2}{r^2\pi^2}\right|.$$

* The absolute value of the expression under the logarithm has been taken to avoid having to work with logarithms of negative numbers.

Or we may take $0<x<\pi$ in the first instance, and then extend the result.

† The above footnote applies also here.

It follows * that $$|\sin x| = x \prod_1^\infty \left|1 - \frac{x^2}{r^2\pi^2}\right|. \qquad \text{(9)}$$

Now, if $2s\pi < x < (2s+1)\pi$, where s is a positive integer,

$$|\sin x| = \sin x, \quad \text{and} \quad \prod_1^\infty \left|1 - \frac{x^2}{r^2\pi^2}\right| = \prod_1^\infty \left(1 - \frac{x^2}{r^2\pi^2}\right).$$

And, if $(2s+1)\pi < x < 2(s+1)\pi$,

$$|\sin x| = -\sin x, \quad \text{and} \quad \prod_1^\infty \left|1 - \frac{x^2}{r^2\pi^2}\right| = -\prod_1^\infty \left(1 - \frac{x^2}{r^2\pi^2}\right).$$

Therefore from (9) we have

$$\sin x = x \prod_1^\infty \left(1 - \frac{x^2}{r^2\pi^2}\right). \qquad \text{(10)}$$

In the same way, from (2), we have

$$\cos x = \prod_1^\infty \left(1 - \frac{2^2x^2}{(2r-1)^2\pi^2}\right). \qquad \text{(11)}$$

These results (10) and (11) hold for all real values of x.

7. *The Infinite Products for sinh x and cosh x.*

In (2) of § 5, put $x = ae^{2\theta}$.

Then we have $$\frac{\sinh(2n+1)\theta}{\sinh\theta} = 2^n \prod_1^n \left(\cosh 2\theta - \cos\frac{2r\pi}{2n+1}\right)$$

$$= 2^{2n} \prod_1^n \left(\sin^2\frac{2r\pi}{2n+1} + \sinh^2\theta\right).$$

And $$\frac{\sinh(2n+1)\theta}{(2n+1)\sinh\theta} = \prod_1^n \left(1 + \frac{\sinh^2\theta}{\sin^2\dfrac{r\pi}{2n+1}}\right).$$

Similarly $$\frac{\cosh(2n+1)\theta}{\cosh\theta} = \prod_1^n \left(1 + \frac{\sinh^2\theta}{\sin^2\dfrac{2r-1}{2n+1}\dfrac{\pi}{2}}\right).$$

Put $(2n+1)\theta = x$, and we have

$$\frac{\sinh x}{(2n+1)\sinh\dfrac{x}{2n+1}} = \prod_1^n \left(1 + \frac{\sinh^2\dfrac{x}{2n+1}}{\sin^2\dfrac{r\pi}{2n+1}}\right),$$

with a corresponding expression for $\dfrac{\cosh x}{\cosh\dfrac{x}{2n+1}}$.

Taking logarithms and using Tannery's Theorem, we find, as in § 6,

$$\sinh x = x \prod_1^\infty \left(1 + \frac{x^2}{r^2\pi^2}\right)$$

and $$\cosh x = \prod_1^\infty \left(1 + \frac{2^2x^2}{(2r-1)^2\pi^2}\right).$$

These results hold for all real values of x.

H. S. Carslaw.

* If $\lim_{n\to\infty} \log\phi(n) = \log a$, we know that $\lim_{n\to\infty} \phi(n)$ exists and is equal to a.

THE TRIANGULAR BILLIARD TABLE PROBLEM.

By C. V. Boys, F.R.S.

A ball is placed anywhere on a billiard table made in the form of an acute-angled triangle. It is required to find a point in a cushion such that if the ball is driven from its initial position to that point and then rebounds once each from the two other cushions it shall return to the point in which it hit the first cushion. It is understood that optical laws of reflection are obeyed and that the balls are points.

Construction.

Let ABC be the acute-angled triangle and P the point within it. Let BC be the side first hit.

From B draw the line BC', a reflection of BC in AB.
" C " " CB', " " " AC.

Find also the points P_1 and P_2, the reflections of P in AB and of P_1 in BC'.
" " " P_3 " P_4, " " " AC " P_3 " CB'.

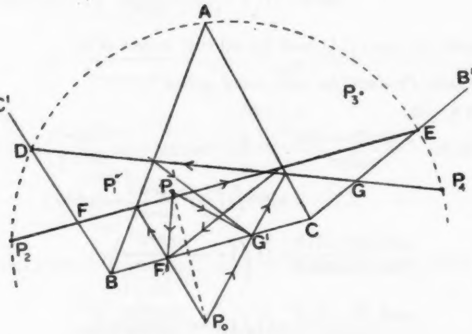

Fig. 1.

Draw an arc of a circle through the 3 points P_2, A and P_4 and cutting BC' in D and CB' in E.

Join EP_2 and DP_4.

Then the portions of these two lines where they cross the triangle are on the tracks of the balls, the first in the clockwise circuit and the second in the anticlockwise circuit.

If these lines cut BC' in F and CB' in G, then the points F', G' in the base BC equidistant from B and C with the points F, G are the points in the base first hit on the clockwise and anticlockwise circuits and the two circuits can be filled in directly as in the figure.

Note. As there are three sides two solutions can in general be found for each side or six in all, but if all six solutions are drawn in one figure, there are so many lines that they cannot be followed.

Proof.

The first shot is to be from P to BC, and take the case of clockwise movement. Construct the two ghost $\triangle$s $C'AB$, $B'AC$ reflections of $\triangle ABC$ and find the ghost points P_1, P_2, P_3, P_4. Then a straight line from P_2 across the three $\triangle$s cutting the base CB' in E will give the one real and the two ghost tracks if the point E is suitably chosen in CB'.

Seeing that the two ghost $\triangle$s are identical and $C'B$, CB' identically correspond, it will be seen that CB' is merely $C'B$ taken round A as a centre through twice the $\angle$ at A.

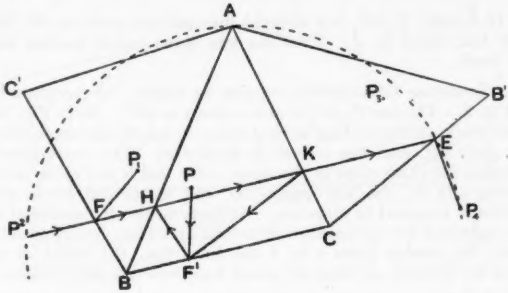

FIG. 1A.

Seeing also that the real track terminates in the same point F' in BC, in which it first hit BC, the ghost distance FB must $=$ the ghost distance EB' and the position P_2 relative to $C'B$ is identical with the position of P_4 relative to CB'.

$$\therefore \angle P_2AP_4 = 2\hat{A}.$$

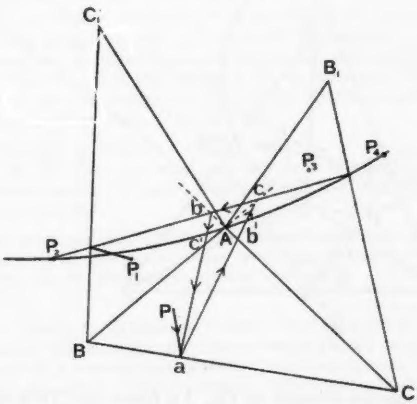

FIG. 2.

If therefore a circular arc P_2AP_4 be drawn, all angles in that arc subtended by P_2P_4 will also $=2\hat{A}$.

$$\therefore \angle P_2EP_4 = 2\hat{A};$$

$\therefore$ P_2F in being moved to the position P_4E has been turned through $2\hat{A}$ which was required.

E is the only point in CB' of which this is true,

thus $EB'=FB$ and $C'F=CE$ and the ghost tracks and the real track correspond.

Similarly for the other way round.

Note. If $\hat{A}=90°$, P_2AP_4 is a straight line and the point in BC first hit will reflect the ball direct to A. Thus the two ways round become an identical there and back.

If the $\hat{A}$ is obtuse the solution remains as before, but becomes imaginary. Refer to Fig. 2. The arc P_2AP_4 is now convex to BC. As in Fig. 1*a*, starting from P the track corresponding to that from P_2 meets the three sides in order $BC \,.\, AB \,.\, AC \,.\, BC$, and the circuit is clockwise. The corresponding track in Fig. 2 takes the three sides in the same order and is clockwise in that sense. After leaving a in BC the ball ignores AC but meets AB produced in c, but the reflection is reversed in direction. It then meets AC produced in b where again it is reflected but reversed in direction and then ignoring AB it comes back to a. In passing from c to b the lines CA, BA which it crosses do not need to be ignored, as they are ghost lines and not really there.

In Fig. 3 $\hat{A}$ is increased to 120°, and in this case the tracks both ways round are shown—as from P_2 in thick lines and as from P_4 in thin lines to reduce the confusion. In consequence of $\hat{A}$ being larger the arc through P_2AP_4 is of smaller radius or is more convex to BC.

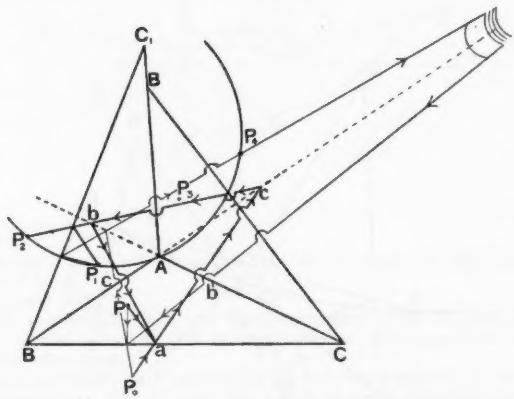

FIG. 3.

If the obtuse-angled triangle of Fig. 2 is taken and AC is the first side hit, so that B becomes the vertex and the figure is drawn changing the letters in this sense, then it depends where the point P is placed, whether the two usual tracks of an acute-angled triangle exist or whether one of them becomes slightly imaginary. The latter case is illustrated in Fig. 4.

In this case the clockwise circuit is from P_2 and is slightly imaginary as it only meets AC by ignoring the base BC and thereafter returns to the point first hit on the wrong side of the base. The anticlockwise circuit as from P_4 is normal. If the point P is not very near B both circuits are normal. It will be seen that these imaginary circuits are too difficult for real billiard players who must therefore be imaginary also.

On further consideration I think that there is no reversal of direction at the points of reflection from the sides produced, but that the track is on the line given by ordinary reflection. The ball then proceeds along this track

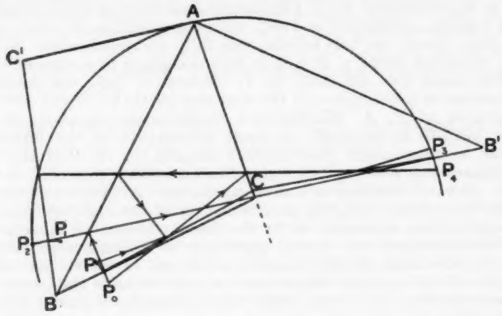

FIG. 4.

to ∞ and comes back to the second point of imaginary reflection from $-\infty$ and the only portion of the infinite track not traversed is that between the sides produced. Clearly the player unless imaginary would become tired of waiting for the ball to return. C. V. BOYS.

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740. First I was greatly taken with natural philosophy; which, while I should have given my mind to logic, employed me incessantly. This I call my *furor mathematicus.* But this worked off, as soon as I began to read it in the college; as men, by repletion, cast off all they have eaten.—Burke to Shackleton [quoted, *Athenaeum*, 1844, p. 612].

741. *Paris Academy of Science.* June 3, 1844.

A paper by M. Wronski, on what he calls true spontaneous locomotion, excited the astonishment of some, and the hilarity of other members.—*Athenaeum*, June 15, 1844.

742. None of us came back [from the Acropolis] wholly unchanged, and the physical beauty of the place alone would have moved even a mathematician's heart.—J. F. Roxburgh, *Modern Odysseys*, p. 2 [per Miss H. P. Hudson, O.B.E.].

743. There is much in Hutton['s Course of Mathematics for the Royal Military Academy] which could only have been received as a point of duty, on the authority of a commanding officer: we, had it been so presented to us, should have demanded a written order.—*Athenaeum*, 1846, p. 148.

744. He [Thomas Kerigan, R.N., F.R.S.] sets us a problem because, as he says, he doubts our mathematical ability. His problem is as follows: Given the Sun's horizontal parallax 8″·613; the Earth's radius 20,898,700 feet; the mass of the Earth 354,936 that of the Sun; and the force of terrestrial gravitation 16·1069 feet *in one second*; to determine the correct length of the sidereal year.

We give in: we cannot determine the *correct* length of the sidereal year upon the supposition that the *Earth is hundreds of thousands of times as heavy as the Sun*, and terrestrial gravity only 16 feet odd in a second.—*Athenaeum*, 1846, p. 359.

REVIEWS.

Lehrbuch der Combinatorik. By E. NETTO. Zweite Auflage. Pp. viii + 341. 1927. (B. G. Teubner, Leipzig and Berlin.)

This is a reproduction by "Photomechanisches Gummidruckverfahren" of the first edition, published in 1901, with the addition of notes and of two fresh chapters. Since the first edition was fully reviewed in the *Mathematical Gazette*, II. (October 1902), p. 216, it is only necessary to comment on the new matter. The notes (pp. 309-338), by T. Skolem, of Oslo, are devoted partly to simplifications of proofs given in the text and partly to recent developments. The appearance of P. A. MacMahon's *Combinatory Analysis* (2 volumes, Cambridge, 1915-6) is naturally of great importance in the latter respect; in particular we may notice that Netto's remark (p. 74) that the treatment of Latin squares has not yet left "das Stadium der Spielerei" is no longer justified in view of MacMahon's investigations. Another matter in which progress has been made is in the generalisation of the eight queens problem; here the appropriate reference is to the later editions of Ahrens: *Mathematische Unterhaltungen und Spiele*, especially to Pólya's work, therein contained, on the so-called doubly-periodic solutions, in which the chess-board is supposed to be infinite and the problem is so to arrange the queens that any square board, of side n, taken out of the whole, contains n queens which cannot take one another.

The two additional chapters are by Viggo Brun and T. Skolem respectively. The first deals with the "distribution-function," defined rather more generally than that which MacMahon introduced with so much skill and profit. Brun's function is

$$D\ (s_0 \ldots s,\ r_0 \ldots r,\ \ldots,\ w_0 \ldots w\ ;\ \sigma_0 \ldots \sigma,\ \rho_0 \ldots \rho,\ \ldots,\ \omega_0 \ldots \omega),$$

the number of distributions of balls of various colours (black, red, ..., white) among distinct urns, in which the number of black balls is between s_0 and s (inclusive), and similarly for the other colours, and the first urn may contain any number of balls between σ_0 and σ, and so on for the other urns. This chapter is written so as to be independent of the earlier part of the book.

Skolem's chapter is devoted to groupings of different objects into systems, which may have objects in common; for example, Steiner's arrangement of 7 symbols in threes such that any two occur together in one and only one three. Skolem makes much use of "pair-systems," and of the result that to every arrangement of n objects in m systems there corresponds dually an arrangement of m objects in n systems, both being capable of treatment as the same pair-system of $m+n$ objects. For example, the arrangement of 4 objects a, b, c, d in 3 systems

$$(abc),\ (abd),\ (acd),$$

which may be denoted by α, β, γ, can be regarded as the pair-system

$$a\alpha,\ a\beta,\ a\gamma,\ b\alpha,\ b\beta,\ c\alpha,\ c\gamma,\ d\beta,\ d\gamma,$$

and then, interpreting a, b, c, d as systems and α, β, γ as objects, we get the arrangement

$$(\alpha\beta\gamma),\ (\alpha\beta),\ (\alpha\gamma),\ (\beta\gamma).$$

A pair-system may be represented geometrically, the objects by points and the pairs by lines joining them, and thus we get to a matter of analysis situs (see Dehn and Heegaard: *Encykl. Math. Wiss.*, III. AB 3, pp. 171-178).

The book is well produced, as one would expect from the publishers; we have only noticed a few minor misprints. But they have bad luck with the name Ramanujan; on p. 146, footnote, is "Romannjan," and on p. 338, Berichtigungen, we have "statt Romanujan lies Ranamnjan." However, they have got it right in the index.

One would have been greatly interested to have read Major MacMahon's comments on this second edition; the book was in fact sent to him for review on its appearance, and it is only his illness and recent death, on Christmas Day, 1929, that has deprived readers of the *Gazette* of an authoritative opinion from the unrivalled master.

F. P. W.

[The following passage from the last letter received by the editor (in August 1929) from Major MacMahon is of pathetic interest:

I have succeeded in getting together from Cambridge certain books to enable me to review for you and Teubner the second edition of Netto's *Combinatorik*, and I was yesterday getting to work—when alas my medical man has forbidden me to continue the work. . . . I am very sorry because I think that I am the proper person to review the book, since I (when well) am in closer touch than anyone else. . . . I have hardly been out of bed or house during about eleven months, and they are afraid of a new break-down without the greatest care. I am really very sorry. . . .]

Cambridge Five-Figure Tables. By F. G. Hall and E. K. Rideal. Pp. viii + 76. 3s. 6d. 1929. (Camb. Univ. Press.)

As a rule it is impossible to see any reason for the existence of a new book of elementary tables; this volume is the welcome exception. Subtraction has its most familiar form, and is therefore most likely to be performed quickly and correctly, if the smaller number is vertically below the larger. Therefore, the compilers suggest, for ready interpolation, consecutive entries should be in the same column, and the order of the arguments should be determined by the condition that the function is to decrease. This principle implies a complete rearrangement of the ordinary tables, and we are given logarithms from 9999 to 1000, and natural and logarithmic sines and tangents at intervals of 1′ from 89° 59′ to 0° 0′; there is also a table of the transformation from radians to degrees and of the values of e^x and e^{-x} at intervals of 0·01 from 1·20 to 0·01, and this table illustrates the obvious difficulty that the principle may be inapplicable if two functions are to be tabulated together: decreasing values of e^x imply increasing values of e^{-x}. The printing throughout is clear, but the type chosen is monotonous. The entire absence of catch entries is unpardonable.

It cannot be said that in the logarithm table the arrangement by decreasing values is given a fair chance. The entries are in framed blocks of 100, the fourth digit running from 9 to 0 downwards according to principle, and the third digit running from 9 to 0 across the page from left to right. A central vertical rule bisects the block, but there is no other subdivision by rule, type, or spacing, and the fourth digit of the argument is printed in a column on the extreme right of the page. The opportunity of error in picking from a uniform column of 10 entries the figures corresponding to an argument more than half a page distant is unreasonably great.

To the trigonometrical table this criticism does not apply. There a degree occupies a column, framed into blocks of 15 minutes, and each block is further divided into three sections by two spaces; moreover the argument is printed in each block. Thus the utmost task is to pick log tan x from a column of 5 entries across the three columns which give sin x, log sin x, and tan x. It is true that by printing the argument centrally, with sin x and log sin x on its left and tan x and log tan x on its right, this task might have been simplified, but even as the figures stand the task is negligible compared with that presented by the logarithm table. In fact, the compilers are successful where they apply their principle wholeheartedly, and the application to simple logarithms has still to be made.

The volume has two pages of physical constants, and a table of atomic weights and isotopic mass-numbers, and ends with four pages of differentials and integrals, where certain familiar mistakes are again to be found.

Standard Table of Square Roots. By L. M. Milne-Thomson. Pp. xii + 91. 7s. 6d. 1929. (Bell.)

The compiler's name is a guarantee of the practical excellence of these tables, which give the square roots of x and $10x$ to eight significant figures for values of x from 100 to 1000 at intervals of 0·1, with first differences; the second differences nowhere exceed 9, and there is a phase-table by means of which the correction for second differences can be made mentally at an adjustment of a first difference.

Using a bold type for arguments and a small type for differences, which also are on the level of the spaces between the entries which they separate, Mr. Milne-Thomson dispenses altogether with "rules," that is, continuous lines, and his pages have a remarkably and pleasantly clean appearance. In brief, the tables are a model of printing, and it is to be hoped that they will exert a wide influence on compilers.

Severely practical, Mr. Milne-Thomson has not swollen his volume by any account of his predecessors. He can hardly have undertaken such a labour as this before acquiring some knowledge of existing tables, and a bibliographical article based upon his notes would, I am sure, be welcomed by many readers of the *Gazette*.

Premières Leçons de Géométrie Analytique et de Géométrie Vectorielle. By E. Lainé. Pp. iv + 47. 4 fr. 1929. (Vuibert.)

After the example set by Appell in the Collection Payot, one hoped that the tyranny of rectangular axes in introductory work on vectors and coordinates was broken: oblique axes express the fundamental idea of decomposition, and although the fear of heavy algebra may explain the use of rectangular axes in a protracted investigation, as long as we are concerned only with first principles and our algebra is in any case trivial, oblique axes are preferable, there being with them no possibility of confusion between decomposition and projection.

M. Lainé's booklet is disappointing, not only because tradition, or possibly a syllabus, has been too binding for him, but because within his own range he has not conveyed a hint of the power which the use of vectors adds in the handling of coordinates. For example, in the course (p. 29) of calculating the distance from the point A to the line D, given by $ax+by+c=0$, we drop the perpendicular AB from A on D. Then:

"The angular coefficient of D is $-a/b$, that of AB is $(y_B-y_A)/(x_B-x_A)$; these two lines being perpendicular, we have

$$-\frac{a}{b}\cdot\frac{y_B-y_A}{x_B-x_A}+1=0,$$

or

$$\frac{x_B-x_A}{a}=\frac{y_B-y_A}{b}.$$

If we denote by ρ the common value of these two ratios, we have

$$x_B-x_A=a\rho,\quad y_B-y_A=b\rho\;\ldots\text{."}$$

This is all incredibly childish. What good has it done us to define—I dare not say to learn about—the scalar product if we fall back on angular coefficients when we want to write down a condition of perpendicularity? Moreover, in the section from which this quotation is made, no condition of perpendicularity should have been wanted. A few pages earlier we have been told that if the equation of a plane is $Ax+By+Cz+D=0$, then (A, B, C) is a vector normal to the plane. Given that (a, b) is some vector perpendicular to the line D, it is the essence of the subject that the vector of the step AB is a multiple of (a, b): the lines of ugly and unnecessary algebra by which he leads up to a pair of scalar equations which he ought to have written down at once are a measure of the failure of an author whose other writings have earned praise.

E. H. N.

Advanced Mathematics for Students of Physics and Engineering. By D. Humphrey. Part I. Pp. viii + 120, 6s. Part II. Pp. viii + 175, 7s. Complete, 12s. 6d. 1929. (Oxford University Press: Humphrey Milford.)

It is a difficult matter to give to advanced students of engineering the type of mathematics that they need. The ground to be covered is extensive, and yet the total amount of time that can be devoted to it is limited, for after all mathematics is only a part of the complete engineering course. Thus, for example, in the London University B.Sc. Engineering Examination, pass students must take at least five subjects and honours students at least six. Mathematics counts as only one of these five or six subjects, yet the mathe-

matical syllabus includes Determinants, De Moivre's Theorem, Calculus (with partial differentiation, also double and triple integrals), Differential Equations, Fourier's Series, Spherical Trigonometry, and Solid Analytical Geometry (up to the tangent planes and normals to surfaces), with applications to Dynamics, Statics, Hydrostatics and Thermodynamics. These branches of pure and applied mathematics are generally treated in separate books, but the attempt to study about nine different mathematical books in one-fifth or one-sixth of the student's total available time is almost foredoomed to failure. Hence teachers will be grateful to Mr. Humphrey, who has tried to combine most of what is needed, at least on the side of pure mathematics, in one text-book. It is a pity that he has so little about the applications, but he has deliberately omitted these, from a feeling that most books on Practical Mathematics obscure the actual mathematical work by a superabundance of worked examples of a practical nature. There is certainly some ground for this opinion. At least one text-book on this subject consists almost entirely of worked examples. The weak point of teaching which is limited in this way is that the student is liable to be baffled as soon as he is faced with an example which differs even slightly from his model. In condemning this tendency Mr. Humphrey is in accordance with our Association's Report (*The Teaching of Mathematics to Evening Technical Students*, 1926). However, the Report explicitly asked that the mathematical work should be correlated with the technical applications, and Mr. Humphrey seems unduly optimistic in his remark that "once he (the student) has become really familiar with the mathematical processes involved in integration, the solution of differential equations, etc., he should have little difficulty in applying them to cases which arise in practice."

Naturally the compression of so many different branches of mathematics into a single book has the consequence that the treatment of any one branch must be very slight. Perhaps the engineering student will consider this more of a merit than a fault, for he has not time to go at all deeply into details. The fullest and best section of the book is that dealing with differential equations. We notice that here, as also in the chapter headed "Applications to Geometry," the author abandons the principles that he had professed in the preface and gives some interesting and useful references to applications. It is rather surprising to find the complete primitive of Clairaut's Form without any mention of the singular solution. However, apart from a few obvious slips (such as those on pp. 1 and 11), the book as a whole seems well adapted for the purpose for which it has been written, which is primarily the fourth and fifth years of evening study in preparation for what are known as the Advanced National Certificates in Mechanical and Electrical Engineering.

H. T. H. Piaggio.

Trigonometry. By A. W. Siddons and R. T. Hughes. Pp. vi+416. Part I, 1s. 9d. Part II, 2s. 6d. Part III, 1s. 9d. Part IV, 3s. 6d. Parts I-II, 3s. 6d. Parts I-III, 4s. 6d. Parts III-IV, 4s. 6d. Parts I-IV, 7s. 6d. 1928-29. (Cambridge University Press.)

The early parts of this book are concerned with I, Numerical Trigonometry, II, Algebraical Trigonometry. An index and answers to exercises are provided.

Parts I and II are very satisfactory. In addition to the bookwork, many common-sense hints are given: *e.g.* in that case of the solution of triangles where three sides are given, "the largest angle should be found by the cosine rule. A second angle can then be found by the sine rule without ambiguity: if the largest angle were obtained by the sine rule, we should not know whether the angle were acute or otherwise." Particularly clear also is the treatment of the ambiguity of sign in "sub-multiple angles." The illustrative examples are abundant, well chosen and completely worked out. The book has a very large number of exercises, bearing more directly than is usual in Trigonometry books on the pupil's work in other branches of Mathematics and Physics, *e.g.* Mechanics, Calculus, Coordinate Geometry and Light; while work in three dimensions is common throughout. The exercises are classified and the

various groups have clear headings, but they are so numerous that a selection will have to be made.

Parts III and IV remind one strongly of the classical "Loney's Part II," but here Prof. Carslaw tells a different and a sadder tale. N. M. G.

The concluding parts of Siddons and Hughes' Trigonometry deal with Imaginary and Complex Numbers, De Moivre's Theorem, the Sine and Cosine as Exponential Functions, Hyperbolic Functions, Finite Series and Finite Products, Infinite Series and Infinite Products. In the Preface the authors state that "the complete book will take a pupil well up to the standard of University Scholarship Examinations, and leave him ready to read advanced treatises." They add that "no attempt has been made to tackle all the difficulties in the more advanced portions, but these have always been pointed out where they exist, and although much is left for the pupil to amplify, it is hoped that there is nothing which he will have to forget."

Unfortunately this hope is ill-founded. It is disquieting to find so much which the student "*will* have to forget" in a school text-book published by the Press of a University which has done so much in recent years to encourage the development of the processes of analysis on a rigorous basis. Any teacher may be excused for thinking that the University gives its imprimatur to all its publications other than encyclopaedias.

For example, the authors have curious ideas on Infinite Series. On p. 252 we read: "We shall assume that the result of differentiating a series is equal to the result of differentiating its sum (this is true provided the series is convergent)." And on p. 304, § 5 we read: "if $S(z)$, the sum of a series involving z, is absolutely convergent over any range of values, we shall make the following assumptions. (i) The result of differentiating the series term by term gives a series whose sum is equal to the result of differentiating $S(z)$, *provided that the derived series is absolutely convergent*,* and provided z is limited to the range of values mentioned above. (ii) The result of integrating the series term by term gives a series whose sum is equal to the result of integrating $S(z)$, provided the range of the integration in each case is within the limits mentioned above."

And at different places in the book the reader is referred to the above passages. As, for instance, on p. 315, § 4: "The sum of a series can sometimes be found by differentiating or integrating a known series and its sum. For the conditions under which this is justifiable see Chapter XVIII, § 5, p. 304."

The authors do not mean the statements to apply only to Power Series. The assumptions are not justified and the statements are, of course, not true.

This work also raises the question of what the pupil reading Higher Trigonometry at school should include in his course. Siddons and Hughes consider that "a knowledge of the elements of Complex Numbers and Series is necessary for the serious Science and Engineering students." And they state that "the study of Infinite Series and of trigonometrical functions of a Complex Variable must be taken by the Mathematical Specialists and the Physicists." Throughout these two parts of their book they use $\sin(x+iy)$ and $\log(x+iy)$ with the same freedom as they use $\sin x$ and $\log x$, where x is real.

With these opinions the reviewer wholly disagrees. Even with the elementary functions, the theory of functions of a complex variable is on quite a different plane from work with the real variable. It may be suitable for the better mathematical specialists. As a topic for an essay question it may occur in Entrance Scholarship Papers. But for such mathematical specialists the advanced text-books provide what is unobjectionable, even if it be hard. A tentative and incomplete treatment (and this is all that is given on the subject in the text) can be of little value. "The serious Science and Engineering student" is already too prone to juggle with symbols which mean nothing to him. The theorems of Analytical Trigonometry—at least all that are fundamental—can be obtained when the real variable only is used.

The reviewer does not mean that De Moivre's Theorem, with its many applications, should be cut out of the course. To prove this theorem only the

* The italics are in the text.

elementary ideas of Algebra are required. The real and imaginary parts are equated in such an equation as $A+iB=C+iD$, where A, B, C and D are real.

The Power Series for $\sin x$ and $\cos x$ can be proved quite rigorously with no great difficulty. See Bromwich's *Infinite Series* (2nd ed.), §§ 59 and 60. But Bromwich in § 60 pointed out that the "proof" which starts with the assumption that $\sin x$ and $\cos x$ can be expressed as Power Series is not logically complete. Siddons and Hughes, on the other hand, consider this "proof" quite sound. (*Cf.* p. 329, § 2.)

From these series it is, of course, true that the relation $e^{ix}=\cos x+i\sin x$ can be deduced, where e^{ix} stands for the exponential series $\sum_0^\infty \frac{(ix)^n}{n!}$; and this relation is often useful in summing series. When it is used, the pupil should be warned of the narrow ledge that he is walking on, and he ought to be told not to wander too far from the usual track.

In Finite Series, the Applications of De Moivre's Theorem, and Finite Products, the book contains many interesting examples and much useful bookwork. And the same is true of the sections on the Hyperbolic Functions. But the reviewer does not like the chapter on Infinite Products. He thinks that, if the authors had known how simply the expressions for $\sin x$, $\cos x$, $\sinh x$ and $\cosh x$ as Infinite Products can be obtained from Tannery's Theorem, they would have given quite a different treatment.

There is a good collection of examples at the end. But in its present form the book seems to the reviewer to be almost dangerous; its excellencies are discounted by its misstatements of fact.

H. S. C.

Plane Trigonometry. By CARL A. GARABEDIAN and JEAN WINSTON. Ed. I. Pp. xviii + 306. 11s. 3d. net. 1929. (Maple Press Co., York, Pa. Copyright by McGraw Hill.)

This book covers the ground of a first year's course, up to the end of solutions of triangles, in the following order:

Angles and trigonometrical functions, variation of the functions by line representation, contracted methods and use of five-figure natural tables, graphs of the functions, circular measure, five-figure logarithms and solutions of right-angled triangles, identities, ratios of obtuse angles, projection, addition and half-angle theorems, sum and product formulae, inverse functions, solution of triangles.

It is not in an elementary book that one looks for a style which, in places, exacts such attention from the reader as this does.

As an example, we find on page 2,

(c) "*Angles are 'equal' if they have the same magnitude and sense. Angles are 'congruent' if they are equal or else differ only by one or more complete revolutions.*"

(d) "*Angles which have the same initial line and the same terminal line are called 'coterminal.' Coterminal angles are congruent; congruent angles become coterminal when their initial lines are made to coincide—as, for example, when they are placed in standard position with vertices at the same point.*"

Other minor points might be criticized.

There are used, for instance, many words which, to say the least, must sound strange to English ears—such words as *cofunction, colog, summand, radicand.*

There is also the usual confusion between magnitude and numerical quantity; thus on p. 29, we have "$\tan\theta = AT/OA = AT/1 = AT$," and on p. 105, "$v=r\omega$, *where* ω *is the angular velocity in radians per unit of time.*"

The notation 2^r, which is used for 2 radians, is open to obvious criticism.

By introducing the concept of a "*slim right triangle,*" and by quoting the "*Pythagorean theorem,*" as, "*the square of the hypotenuse is equal to the sum of the squares of the legs,*" the authors strike a new line, which leaves one wondering what the physical disabilities of the hypotenuse really are!

While trivialities are often treated "in extenso," work of importance is either omitted altogether or is relegated to the position of examples.

No definition of a limit is given, although there is an explanation of convergence, whilst $lim \sin\theta/\theta$ is dismissed with the bald statement that it is obviously 1.

The main feature of the book consists in the obtaining of results to a desired degree of accuracy, a feature which may render the book of use to those who intend to become computers.

In the solution of triangles, Newton's expression

$$(a+b)/c = \cos\tfrac{1}{2}(A-B)/\sin\frac{C}{2}$$

is used as a check formula.

The part of the work dealing with graphs is fairly well done; the graphs are beautifully printed in orange and black.

The general "lay out" and printing are both excellent, but the price is too high for the material offered.

In conclusion: the book is definitely American, and may meet successfully the needs of American schools. It would not, however, be suitable as a text-book in English schools, although it might find a useful place in the school library. We note the confidence expressed by the *Edition I.* on the title page.

V. N.

The Aims of Mathematical Physics. *An Inaugural Lecture delivered before the University of Oxford,* 19*th November,* 1929. By E. A. Milne. Pp. 28. 2s. net. 1929. (Milford; Oxford University Press.)

It was the wish of the founder of the Rouse Ball Chair of Mathematics at Oxford that its occupant should treat from time to time of the historical and philosophical aspects of the science. For the first lecture "the aims of mathematical physics" was selected as an appropriate subject.

"There is one distinguished mathematical physicist whom his physical friends describe as a mathematician and his mathematical friends describe as a physicist. The truth of the matter is that he is neither the one nor the other, but a separate kind of being, namely, the mathematical physicist."

The lecturer very clearly demonstrated the need for the existence of such a being and his peculiar functions in the pursuit of his aim, and laid stress on the qualifications necessary to the successful performance of his *rôle*. He then proceeded to discuss the manner in which developments are being made, and concluded a most attractive discourse with advice to those investigators who are especially interested in astrophysics.

GIRLS' SCHOOLS COMMITTEE.

The Girls' Schools Committee have decided to study and experiment on types of papers to test (1) the mathematical ability of candidates taking Entrance Scholarship examinations to Secondary Schools; (2) the mathematical ability of pupils after 1, 2, 3, 4 years' work in Mathematics; (3) the mathematical ability of pupils other than Scholarship candidates.

The Committee will welcome any help or suggestions from Branches or from individual members of the Association.

E. WISE, Hon. Sec.,

1st February, 1930. 21 Amersham Hill, High Wycombe, Bucks.

ERRATA.

P. 364, vol. xiv. *Gleaning.* *For* 647 *read* 648.

P. 20, vol. xv. *Gleaning* 720. *For* Guyan *read* Guyau.

Vol. xv. p. 18, line 22 up. *For* "m is either r or a factor of r," *read* "N_m is either $m-1$ or a factor of $m-1$."

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AT THE UNIVERSITY PRESS, GLASGOW

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